S. P. Rigaud
Oct. 20. 1826

THE

FLUXIONAL CALCULUS.

AN

ELEMENTARY TREATISE,

DESIGNED FOR THE

STUDENTS OF THE UNIVERSITIES,

AND FOR

THOSE WHO DESIRE TO BE ACQUAINTED WITH THE PRINCIPLES OF ANALYSIS.

BY THOMAS JEPHSON, B. D.

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Quas nos quantitates variabiles vocamus, eas Angli nomine magis idoneo quantitates fluentes vocant, et earum incrementa infinitè parva seu evanescentia fluxiones nominant, ita ut fluxiones ipsis idem sint, quod nobis differentialia. Hæc diversitas loquendi ita jam usu invaluit ut conciliatio vix' unquam sit expectanda; equidem Anglos in formulis loquendi lubenter imitarer, sed signa quibus nos utimur illorum signis longe anteferenda videntur.

Euler, *Calc. Int.* vol. i. p. 3.

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PREFACE.

THIS Work will contain the substance of a course of lectures read to pupils some few years ago. Part of the leisure which I have had since that period has been devoted to such alterations and corrections as may make it more generally useful. It is with this view that I have not confined myself to the Newtonian doctrine of limits, but have introduced the principles of La Grange's Theory of Functions. The two systems meet in Taylor's Theorem, and that being once established, the difference is merely nominal.

I have endeavoured to render this work as independent of all others as possible, and have required as little previous knowledge as the nature of the subject will admit. This consists of the elements of Geometry, Trigonometry, Algebra, and Conick Sections.

The works on these subjects which I have always used, and to which I of course refer, are Simson's Euclid, Professor Woodhouse's Trigonometry, third edition, Dr. Wood's Algebra, sixth edition, and Mr. Peacock's Conick Sections, second edition.

To complete the work, it should be extended to a discussion of certain curves—to a treatise on the calculus of fluents—the integration of fluxional equations—the calculus of variations—the application of the calculus to curved surfaces, including those of double curvature. This, together with its application to such parts of Natural Philosophy as usually form an academical course of lectures, may appear before the publick in a second volume.

CONTENTS.

ERRATA.

Page 24, *for* l. 13, *read* By (7) $dx = -du \cot.u \operatorname{cosec}.u \therefore du = \dfrac{-dx}{\cot.u \operatorname{cosec}.u}$.

27, l. 4, *for* $(1-x^2)^2$, *read* $(1-x^2)^{\frac{3}{2}}$.

29, l. 4, *for* $e^{\tan.x}$, *read* $e^{\tan.^{-1}x}$.

40, l. 7, *for* $-\dfrac{\sqrt{a^4+x^4}}{a^4x^2}$, *read* $-\dfrac{\sqrt{a^4+x^4}}{2a^4x^2}$.

45, l. 7 from bottom, *for* 44″.81, *read* 44″.981.

49, l. 2, *read* Hence $-ly = 1 - y + \frac{1}{2}(1-y)^2 +$ &c.

52, l. 3, insert $-$ after $3.l2$.

63, l. 15 from bottom, *for* $\mathrm{L}(a+1)$, *read* $\mathrm{L}(a \times 1)$.

69, l. 5, *for* x, *read* x^n.

72, l. 2 from bottom, *for* $y^p dy$, *read* $-y^p dy$.

75, l. 5, insert $-\dfrac{\sqrt{a-x}}{ax}$.

99, l. 6 from bottom, *for* $\dfrac{\sqrt{1+\sin.^2x}}{\sqrt{1-\sin.^2x}}$, *read* $\dfrac{\sqrt{1+\sin.x}}{\sqrt{1-\sin.x}}$.

100, l. 12, *for* $(1-y^2)^{\frac{3}{4}}$, *read* $(1-y^2)^{\frac{3}{2}}$.

l. 20, *read* $\therefore$ (2. 54. Ex. 4) $u = \dfrac{x^3\sin.^{-1}x}{3} + \dfrac{\sqrt{1-x^2}(x^2+2)}{9}$.

103, l. 7, *for* $\frac{1}{3}l$, *read* $\frac{1}{4}l$.

147, l. 10 from bottom, *for* $3xy^2f.,'' + y^3f.'_{,,}$ *read* $3xy^2f.'_{,,} + y^3f._{,,,}$

170, l. 17, + &c. is to be moved out of the bracket.

184, l. 9 from bottom, *for* $\dfrac{1}{1.2}$, *read* $\dfrac{1}{2.3}$.

189, l. 4, *for* $\dfrac{du}{dx}$, *read* du.

202, l. 4, *for* $2a^3$, *read* $2a^3x$.

206, l. 2, *for* $b6^2$, *read* $6x^2$.

225, l. 3 from bottom. The chapter here referred to is removed to the second volume.

239, l. 17. The N in this and the two following lines is to be multiplied by *abc*.

257, Praxis 2. This curve has four conjugate points, which are the only points that belong to the curve. The equation may be considered as a particular case of $(x^2-a^2)^2 + (y^2-b^2)^2 = c^4$. The equation of two ellipses whose axes are at right angles to each other is $y^4 = (a^2-x^2)(b^2-x^2)$.

347, l. 5, *for* $\dfrac{1}{2}\sqrt{1-x^2}$, *read* $\dfrac{x\sqrt{1-x^2}}{2}$; also enclose $\dfrac{x^2}{4} + \dfrac{1.3}{2.4}$ in . . . . brackets.

404, l. 15, *for* $(y-w)$, *read* $(y-w)p$.

414, l. 5 from bottom, *for* $\mathrm{R}p$, *read* $\mathrm{R}q$.

THE

ELEMENTS

OF THE

FLUXIONAL CALCULUS.

PART FIRST.

Explanation of Terms.

1. ALL quantity, whether represented by lines and figures, or expressed algebraically, may be considered as generated by motion: lines, by the motion of a point; areas, by a line moving parallel to itself; solids, by the motion of an area along a fixed line, which is the axis of the solid.

When quantity is expressed algebraically, the first letters of the alphabet are usually taken to denote the constant, and the last the variable part.

In curves, the ordinate and abscissa are variable lines; and the diameters, axes, parameters, &c. are, in general, considered as constant quantities.

2. A function of a variable quantity is any expression of calculation whatever into which the variable, mixed or not with constant quantities, enters.

Thus x^n, a^x, log. x, $ax+b$, $\sqrt[3]{m+nx+cx^2}$, $\frac{1+ax+bx^2}{1+\alpha x+\beta x^2}$, sin. x, sin. $x^{\cos. x}$, &c. &c. are all functions of x.

Symbols to represent these are $\mathrm{F}x, fx, \phi x, \psi x$, &c.; thus, if in any particular case, $\mathrm{F}x$ represents $\frac{1+ax+bx^2}{1+\alpha x+\beta x^2}$ and fx

represents $ax+b$, then $Fy = \dfrac{1+ay+by^2}{1+\alpha y+\beta y^2}$ and $fy = ay + b$.

$e^{\frac{x}{a}} + e^{\frac{-x}{a}}$ is a function of $\dfrac{x}{a}$. Also, the quantity $(a + bx + cx^2)^n + bx + cx^2$ may be put under the form of a function of $a + bx + cx^2$, for it equals $(a + bx + cx^2)^n + (a + bx + cx^2) - a$.

The ordinate of a curve is a function of its abscissa. The space described by a body projected downwards is a function of the time of descent, the velocity of projection entering the function as a constant. And, generally, if it can be shown of two quantities m and n, that m varies when n varies, and that m is constant when n is constant, then we know that $m = \varphi n$, where the form of φ is to be determined from the conditions of the question.

If in the symbol fx we make $x = 0$, it becomes $f.$, which, therefore, represents a constant quantity, or rather a function in which x does not enter. Thus, if $fx = ax + b, f. = b$; if $fx = \dfrac{a}{x} + b, f\cdot = \infty$,

When the function is enclosed in a bracket, and no other reason appears, it expresses the function when a particular value is assigned to the variable.

Analytical functions are either *algebraick* or *transcendental.* The former are subdivided into *rational* and *irrational:* the latter, into *exponential, logarithmick,* and *circular.* Instances of transcendental functions are a^x, log. x, sin. x, $a + b$. cos. x, &c.

$x^3 - axy + y^3$ is a function of two variables, and general symbols are $F(x, y)$, $f(x, y)$, &c.

If $y^2 - xy - a^2 = 0$, y is said to be an *implicit* function of x; but when this equation is solved with respect to y, or when we put it under the form $y = \dfrac{x}{2} \pm \sqrt{\dfrac{x^2}{4} + a^2}$, y is then said to be an *explicit* function of x. Symbols for each of these are $F(x, y) = 0$, and $y = fx$.

3. $F(fx)$ represents a function of a function of x; thus, if $fx = (a+bx+cx^2)^n$, $F(fx)$ may represent $(a+bx+cx^2)^{mn}$, or $e^{(a+bx+cx^2)^n}$, &c. &c.

If $F = f$, $f(fx)$ is denoted by f^2x, $f(f^2x)$ by f^3x, &c.; and, according to this notation, sin.^2x does not represent the square of the sine of x, but the sine of the sine of x, *i. e.*

sin. x being formed into an arc, its sine is the line represented by $\sin.^2 x$ *. Also $\log.^2 x = \log.$ of $\log. x$.

We must be careful to distinguish $f^2 x$ from the square of fx or fx^2; thus, let $fx = x^3$, then $f^2 x$, in this notation, $= (x^3)^3 = x^9$, but $fx^2 = x^6$.

4. The variable which is under the sign of the function in $y = fx$ is called the *principal* or *independent* variable; the other the dependent variable, as it depends upon the value we assign to x.

5. If the independent variable be increased or diminished, and the function can be expanded, we can compare the increments of the variables.

As an example, take $y = x^3$, and when x becomes $x + h$, let y become Y, then $\text{Y} = (x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$, therefore $\text{Y} - y = 3x^2h + 3xh^2 + h^3$, and dividing by h, $\frac{\text{Y}-y}{h} = 3x^2 + 3xh + h^2$, which is the value of inc. y : inc. x.

The first term of this result is independent of h; if then we suppose $h = 0$, $\text{Y} = y$, and we have $\frac{0}{0} = 3x^2$, the meaning of which will be explained, Ch. V.

6. In the present work we consider all such functions as flowing quantities or fluents, and the two problems which include the whole of the fluxional calculus are, (1). Given the fluents, to find their rates of increase; and, (2). Given the rates of increase, to find the fluents. The first is called the direct, the second the inverse method of fluxions.

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* We shall not make use of this notation in "circular" functions without giving the reader notice.

CHAPTER I.

On the first fluxion of a function containing one or more variables.

7. *Def.* 1. THE "*limiting ratio*" of two dependent variables is that to which their ratio may be made to approach nearer than by any proposed difference.

Def. 2. The ratio of the *fluxions* of two dependent variables is the limiting ratio of their cotemporary increments.

Cor. Hence, if two variable quantities are always equal, their fluxions must be equal; if they are in a constant ratio, their fluxions must be in the same ratio.

The quantities whose fluxions are considered must be *dependent* quantities, such as the abscissa and ordinate of a curve, or a function and its variable, or two functions whose principal variables depend upon each other.

To find the limiting ratio of the increments, diminish them indefinitely, and their last ratio or the ratio in which they vanish is the ratio of the fluxions.

If this last ratio be finite, it may be expressed by finite quantities, and may be represented by any notation that may be thought best adapted to the purposes of calculation.

We shall, in general, substitute x, or y, or z, for the principal variable, and u for the function, and shall suppose that when x becomes $x + h$, u becomes U; hence if $u = \mathrm{F}x$, it follows from the definition, that the limit of

$$\frac{\mathrm{U} - u}{h} = \frac{\text{flux.}\, u}{\text{flux.}\, x} = \frac{\dot{u}}{\dot{x}} \text{ or } \frac{du}{dx},$$

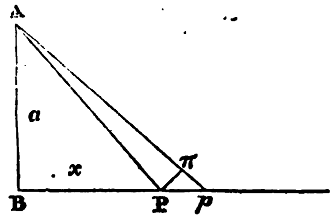
according as we use Newton's or Leibnitz's notation. In the present work we shall adopt the latter.

Though the idea of a fluxion is essentially relative, yet it is sometimes convenient in calculation to consider it as something positive and absolute.

Thus, if $\frac{du}{dx} = 3x^2$, we have $du = 3x^2dx$; in this case,

the fluxion of one of the quantities as x is assumed of some determinate magnitude, and the value of the fluxion of the other quantity is determined from the equation $\frac{du}{dx} = 3x^2$.

We shall conclude this article by the following illustration of the definition.



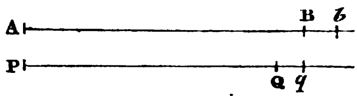
Let ABP be a right angled triangle, whose altitude AB is constant, and base BP variable; and let it be required to compare the fluxion of the base with the fluxion of the hypothenuse.

Suppose BP to become Bp, join Ap, and in it always take Aπ = AP, join Pπ; then Pp and $p\pi$ are the corresponding increments of the base and of the hypothenuse. Now diminish the $\angle$PAp without limit, and the $\triangle$P$p\pi$ being ultimately similar to $\triangle$ABP, we have Pp : $p\pi$, in the limit, = AP : BP, which is the required ratio of the fluxions.

Cor. If we suppose BP to be generated by an uniform motion, its fluxion is a constant quantity, and the fluxion of AP $\propto \frac{BP}{AP} \propto \frac{x}{\sqrt{a^2+x^2}} \propto \frac{1}{\sqrt{\frac{a^2}{x^2}+1}}$, and is therefore a continually increasing quantity.

The ratio of the fluxions is defined by Sir I. Newton to be the ratio of the velocities; and to show that the two definitions agree, the following proposition must be demonstrated.

8. *The limiting ratio of the increments of two flowing quantities is the ratio of their velocities.*



Let AB increase uniformly with the velocity of a, and PQ increase with a variable velocity, which at Q is V.

Let Bb, Qq, be increments generated in the same time t. Also, suppose V to become V $\pm$ v when the generating point is at q, and that V $\pm$ m continued *uniform* would cause the point Q to describe Qq in the time t, then m is manifestly less than v; and Bb : Qq : : a : V $\pm$ m; and since m is less

than v, and v vanishes when t vanishes, the limiting ratio of $\mathrm{B}b : aq$ is that of $a : \mathrm{V}$.

If both velocities are variable, compare each with a uniform velocity, and by an ex æquo the same conclusion will be obtained. (Encyclopædia Britan. Art. Fluxions, p. 722.)

9. A constant quantity has no fluxion. The fluxion of an increasing quantity is positive, and of a decreasing quantity negative, and *vice versâ* if the variable itself be negative.

Thus, as the radius of a circle revolves through its four quadrants, the fluxion of the *sine* is positive in the first and fourth, and negative in the second and third quadrants.

10. *The fluxions of two equal functions of the same variable are equal.*

Let u and v be functions of the same variable x, and when x becomes $x+h$, let u and v become U and V, then since $u = v$ and $\mathrm{U} = \mathrm{V}$, $\mathrm{U} - u = \mathrm{V} - v$, and dividing by h, $\frac{\mathrm{U}-u}{h} = \frac{\mathrm{V}-v}{h}$, and taking these ratios in their limit,

$$\frac{du}{dx} = \frac{dv}{dx} \text{ or } du = dv.$$

If $u : v$ is a constant ratio, that of $n : 1$, then
$\mathrm{U} : \mathrm{V} :: u : v :: n : 1$, $\therefore$ div.° $\mathrm{U} - u : \mathrm{V} - v :: n : 1$; hence (7. Cor.) $du : dv :: n : 1$.

When from a function we derive its fluxion, or in other words, when we derive the ratio which is the limit of $\frac{\mathrm{U}-u}{h}$ from $u = \mathrm{F}x$, we are said to *differentiate* the function. The laws of the derivation are regulated solely by the *form* of the function, and it will be seen in the course of the chapter, that they are wholly independent of the increment assigned to its variable.

We shall now proceed to establish rules for the differentiation of a function, whatever be its form.

11. Rule 1. *Constant quantities united by the signs + or − disappear by differentiation.*

For let $u = x \pm a$, therefore $\mathrm{U} = x \pm a + h$, therefore $\frac{\mathrm{U}-u}{h} = 1$, a constant ratio, and in the limit, $\frac{du}{dx} = 1$, or $du = dx$.

12. Rule 2. *The constant multiplier or divisor is to be retained.*

For let $u = ax$, then $\text{U} = a \cdot (x + h)$, and $\frac{\text{U} - u}{h} = a$, which in the limit gives $\frac{du}{dx} = a$, or $du = adx$, where a may be either whole or fractional.

13. *The fluxion of the sum of any number of functions of the same variable is equal to the sum of their fluxions.*

Or $d \cdot (u + v - w \pm \&c.) = du + dv - dw \pm \&c.$ where u, v, w, &c. are functions of the same variable x.

For let the increment of x be h, and the cotemporary increments of u, v, w, &c.; l, l', l'', &c. respectively: then inc. of $(u + v - w \pm \&c.)$: inc. of $x :: l + l' - l'' \pm \&c. : h$

$$:: \frac{l}{h} + \frac{l'}{h} - \frac{l''}{h} \pm \&c. : 1;$$

and taking these ratios in their limit, we have

$$d \cdot [u + v - w \pm \&c.] : dx :: \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx} \pm \&c. : 1;$$

hence $d \cdot [u + v - w \pm \&c.] = du + dv - dw \pm \&c.$

Rule 3. *Hence, if a function is composed of several terms into which the variable enters, it is differentiated by differentiating each term and prefixing its proper sign.*

14. *Required to differentiate a variable raised to any power.*

Let $u = x^n$, $\text{U} = (x + h)^n =$ (binomial theor.) $x^n + nx^{n-1}h + n \cdot \frac{n-1}{2} x^{n-2}h^2 + \&c.$; therefore $\frac{\text{U} - u}{h} = nx^{n-1} + n \cdot \frac{n-1}{2} x^{n-2}h + \&c.$, h entering into all the succeeding terms.

Now diminish h without limit, and we have $\frac{du}{dx} = nx^{n-1}$, or $du = n \times x^{n-1} \times dx$.

Cor. 1. The same demonstration holds if n be fractional.

Or thus. Required to differentiate $x^{\frac{r}{s}}$.

Let $u = x^{\frac{r}{s}}$, therefore $x^r = u^s$, and differentiating as in the article, $rx^{r-1}dr = su^{s-1}du$, therefore $du = \frac{rx^{r-1}dx}{su^{s-1}}$

$$= \frac{r}{s} \times \frac{x^{r-1}dx}{x^{r-\frac{r}{s}}} = \frac{r}{s} \times x^{\frac{r}{s}-1} \times dx.$$

Cor. 2. Next, let n be negative.

First, suppose $u = x^{m-n}$, therefore $du = (m-n)x^{m-n-1}dx$, in which if $m = 0$, we have $d \cdot x^{-n} = -nx^{-n-1}dx$.

Cor. 3. Required to differentiate a compound quantity, as $(a^m + x^m)^n$.

Let $u = a^m + x^m$, therefore $u^n = (a^m + x^m)^n$, and, differentiating, $d \cdot (a^m + x^m)^n = nu^{n-1}du = n \times (a^m + x^m)^{n-1} \times mx^{m-1}dx$.

Hence, Rule 4. *Multiply by the index, diminish the index by unity, and multiply by the fluxion of the root.*

Examples to the preceding rules.

$u = a + bx \therefore du = bdx.$

$u = x^3 \therefore du = 3x^2dx.$

$u = \frac{2}{3}x^{\frac{3}{2}} \therefore du = x^{\frac{1}{2}}dx.$

$$u = \frac{1}{x} = x^{-1} \therefore du = \frac{-dx}{x^2}.$$

$$u = \frac{1}{x^n} \therefore du = \frac{-ndx}{x^{n+1}}.$$

$$u = \sqrt{2ax + x^2} \therefore du = \frac{(a+x)dx}{\sqrt{2ax+x^2}}.$$

$u = (a^2 + x^2)^3 \therefore du = 3 \cdot (a^2 + x^2)^2 2xdx.$

Or thus,

$u = a^6 + 3a^4x^2 + 3a^2x^4 + x^6.$

$\therefore du = 6a^4xdx + 12a^2x^3dx + 6x^5dx = 6(a^2+x^2)^2xdx.$

$$u = a + \frac{b}{x^{\frac{2}{3}}} - \frac{c}{x^{\frac{4}{3}}} + \frac{e}{x^2} \therefore du = \frac{-\frac{2}{3}bdx}{x^{\frac{5}{3}}} + \frac{\frac{4}{3}cdx}{x^{\frac{7}{3}}} - \frac{2edx}{x^3}.$$

$$u = \frac{1}{\sqrt{1-x^2}} \therefore du = \frac{xdx}{(1-x^2)^{\frac{3}{2}}}.$$

$$u = \frac{a+x}{x} = ax^{-1} + 1 \therefore du = \frac{-adx}{x^2}.$$

$$u = \frac{x}{x+\sqrt{1-x^2}} = (1+\sqrt{x^{-2}-1})^{-1} \therefore du =$$

$$\frac{-d \cdot \sqrt{x^{-2}-1}}{(1+\sqrt{x^{-2}-1})^2} = \frac{(x^{-2}-1)^{-\frac{1}{2}} \cdot x^{-3}dx}{(1+\sqrt{x^{-2}-1})^2} = \frac{dx}{\sqrt{1-x^2}(x+\sqrt{1-x^2})^2}$$

$$u = \sqrt[m]{\frac{a+x}{\sqrt[m]{\frac{a+x}{\sqrt[m]{\frac{a+x}{\&c.}}}}}} \therefore u^m = \frac{a+x}{\sqrt[m]{\frac{a+x}{\sqrt[m]{\frac{a+x}{\&c.}}}}} = \frac{a+x}{u} \therefore u^{m+1} = a + x,$$

$$\therefore u = (a+x)^{\frac{1}{m+1}} \therefore du = \frac{1}{m+1} \cdot (a+x)^{-\frac{m}{m+1}} dx.$$

$$u = \sqrt{a+bx+cx^2} \therefore du = \tfrac{1}{2} \cdot (a+bx+cx^2)^{-\frac{1}{2}} \cdot (bdx + 2cxdx)$$
$$= \frac{\frac{1}{2}bdx + cxdx}{\sqrt{a+bx+cx^2}}.$$

$$u = \sqrt{x^2 + \sqrt{a^2+x^2}}$$
$$\therefore du = \tfrac{1}{2}\left[x^2 + (a^2+x^2)^{\frac{1}{2}}\right]^{-\frac{1}{2}} \left\{2xdx + \frac{xdx}{(a^2+x^2)^{\frac{1}{2}}}\right\}$$
$$= \frac{xdx}{\sqrt{x^2 + \sqrt{a^2+x^2}}} + \frac{\frac{1}{2}xdx}{(a^2+x^2)^{\frac{1}{2}}\sqrt{x^2+\sqrt{a^2+x^2}}}.$$

$$u = \sqrt[4]{\left[a - \frac{b}{\sqrt{x}} + \sqrt[3]{(c^2-x^2)^2}\right]^3} = \left[a - \frac{b}{x^{\frac{1}{2}}} + (c^2-x^2)^{\frac{2}{3}}\right]^{\frac{3}{4}}.$$

$$\therefore du = \tfrac{3}{4} \cdot \left[a - \frac{b}{x^{\frac{1}{2}}} + (c^2-x^2)^{\frac{2}{3}}\right]^{-\frac{1}{4}} \left\{\frac{\frac{1}{2}bdx}{x^{\frac{3}{2}}} - \tfrac{4}{3} \cdot (c^2-x^2)^{-\frac{1}{3}} xdx\right\}$$

$$= \left\{\frac{\dfrac{\frac{3}{8}b}{x^{\frac{3}{2}}} - \dfrac{x}{\sqrt[3]{c^2-x^2}}}{\sqrt[4]{a - \dfrac{b}{x^{\frac{1}{2}}} + \sqrt[3]{(c^2-x^2)^2}}}\right\} dx.$$

15. Hitherto we have considered the fluxion of a function which contains but one variable. When it contains two, its increment, and consequently its fluxion, must evidently depend upon the increment or decrement of both of them.

There is a difficulty, however, in the principle of differentiating a function such as $u = F(x, y)$, which requires to be noticed.

If we have an equation in which x is the only variable which enters mixed with constant quantities, the value of x is fixed and determined. If the equation contains two variables, by transposition we may put the whole expression u under the form $u = F(x, y) = 0$; and solving this

equation with respect to x, we shall have $x = fy$, which, as will be seen in the course of the chapter, can always be differentiated. By this process we obtain a fluxional equation $dx - d(fy) = 0$, which, replacing y's value, must be identical with $d.\text{F}(x, y) = 0$, since x and y are unaltered. The student may take as an example $u = x^2 - 2ax + y^2 = 0$.

But if we have the function $\text{F}(x, y)$ without any condition annexed, x and y are now independent quantities, and as such we cannot compare their fluxions. Take the simplest function possible, $u = x + y$; let h and k be the increments of x and y, then $\text{U} - u = h + k$, and dividing by k, $\frac{\text{U} - u}{h} = 1 + \frac{k}{h}$; but $\frac{k}{h}$ in the limit does not equal $\frac{dy}{dx}$, unless x and y are dependent quantities. If we have another condition, viz. that $u = x + y = 0$, then taking the ratios in their limit, we have $\frac{du}{dx} = 1 + \frac{dy}{dx}$, and du, which $= 0, = du + dy$, and $dx = - dy$, the increment of x being equal to the decrement of y.

In the following articles in which rules are investigated for differentiating functions of two or more variables, we assume that all the variables may be considered as depending upon some one of them. It is in the spirit of analysis to make any suppositions with respect to *indeterminate* quantities, which are not at variance with each other. It is this which enables the Analyst to give to his calculations all the generality of which they are capable. When, however, he applies these principles to the solution of problems either of algebra, geometry, or physics, he has farther to consider whether the conditions of the question render such suppositions allowable. It is upon this principle that in the equation $u = \text{F}(x, y)$, when x and y are indeterminate, we may suppose one of the variables as y to be constant. When we differentiate u on this supposition, the result is called its *partial* fluxion taken with respect to x. No notation for these partial fluxions has been hitherto proposed that is universally adopted. Fontaine represents them by $\frac{du}{dx}dx$, $\frac{du}{dy}dy$; the first being the fluxion of u taken with respect to x, and the other its fluxion taken with respect to y.

$\frac{du}{dx}$ and $\frac{du}{dy}$ are called the *partial fluxional coefficients* of u.

Thus, if $u = (a + x)^2 \cdot y^3$,

$$\frac{du}{dx}dx = 2y^3 \cdot (a + x)dx \text{ and } \frac{du}{dy}dy = 3 \cdot (a + x)^2 y^2 dy.$$

This mode of differentiating will be a source of embarrassment to the student, unless during the calculation he constantly observes which of the du's represent the total, and which the partial fluxions of u. In practice, a function of more than one variable is almost always differentiated with respect to one of them; so that partial fluxions occur the most frequently; their form prevents any mistake. All then that is necessary, when they occur together, is to place some distinguishing mark upon the *total* fluxion: for my own convenience, I generally place a dot over the d. Thus the equation $\frac{\dot{d}u}{dx} = \frac{du}{dx} + \frac{du}{dy} \ \frac{dy}{dx}$, shows from its form not only that $u = \text{F}(x, y)$, but that y is also considered as a function of x, and the proposition which it expresses, multiplying both sides by dx, is this: the *total* fluxion of u = its partial fluxion taken with respect to x + its partial coefficient taken with respect to y multiplied into the fluxion of y taken with respect to x *.

If we have two equations between x and y, we may differentiate both of the equations partially at the same time, for by adding $\text{F}(x, y) = 0$ and $f(x, y) = 0$, we have an equation $\varphi(x, y) = 0$, which may be partially differentiated; but we must bear in mind that the same quantity is to be considered constant in each equation, otherwise the two hypotheses will be inconsistent.

Further, it is obvious, that so long as x and y are indeterminate, we may make any supposition whatever as to the relative magnitude of their respective increments, provided

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* Euler, in order to characterise the partial coefficient, encloses it in brackets; La Croix and others propose that the total fluxion should be inserted in brackets: but as we use the brackets to denote the state of the function when a particular value is assigned to its variable, either notation would be inconvenient.

that the increment of u is not affected by it. Thus we may obtain the *total* fluxion of u, by assuming either that their increments are equal, or that they are to each other in any assigned ratio; the only condition being, to recur to our first instance, that $\text{U} - u = h + k$.

16. *To differentiate a product of two variables.*

Let $u = xy$, and let u become U in consequence of x becoming $x+h$, and y becoming $y + k$; therefore $\text{U}=(x+h)\cdot(y + k) = xy + kx + hy + hk$, therefore $\text{U} - u = kx + hy + hk$; and to compare the fluxion of u with the fluxion of x, divide by h, and we have $\frac{\text{U}-u}{h} = \frac{k}{h}\cdot x + y + k$, and taking these quantities in their limit, $\frac{du}{dx} = \frac{dy}{dx}x + y$, therefore $du = xdy + ydx$.

The same result would have been obtained, had we found $\frac{du}{dy}$: hence

Rule 5. *The fluxion of the product of* x *and* y *is* y *multiplied into the fluxion of* x + x *into the fluxion of* y.

The fluxion of xy then consists of two parts, the first of which is its partial fluxion taken with respect to x, the other its partial fluxion taken with respect to y.

17. *To differentiate a product of any number of variables.*

$$\begin{aligned} d\cdot xyz = yzdx + xd\cdot yz &= yzdx + x[zdy + ydz] \\ &= yzdx + xzdy + xydz. \end{aligned}$$

Similarly, $d\cdot xyzvw\cdots = dx\cdot yzvw\cdots + dy\cdot xzvw\cdots + dz\cdot xyvw + \&c.$

Hence, Rule 6. *Whatever be the number of factors, the fluxion of the product equals the sum of the products of the fluxion of each factor multiplied by all the other factors.*

Cor. The fluxion of x^n obtained in Art. 14. may also be deduced from this, without the aid of the binomial theorem.

For $\frac{d.x^n}{x^n} = \frac{d(x\cdot x\cdot x\cdots \text{to } n \text{ terms})}{x\cdot x\cdot x\cdots \text{to } n \text{ terms}} = \frac{dx}{x} + \frac{dx}{x} + \frac{dx}{x}\cdots$ to n terms $= \frac{ndx}{x}$, therefore $d\cdot x^n = nx^{n-1}dx$.

(Saunderson's Fluxions.)

18. *To differentiate a quotient.*

Let $u = \frac{x}{y}$, and making the same suppositions as in

Art 17, we have $\upsilon - u = \frac{x+h}{y+k} - \frac{x}{y} = \frac{hy-kx}{y\cdot(y+k)}$, therefore $\frac{\upsilon-u}{h} = \frac{y - \frac{k}{h}\cdot x}{y\cdot(y+k)}$.

Now diminish the increments without limit, and we have

$$\frac{du}{dx} = \frac{y - \frac{dy}{dx}\cdot x}{y^2},\text{ or } du = \frac{ydx-xdy}{y^2}.$$

We should have had the same result if we had differentiated $\frac{x}{y}$ as a product.

For $\frac{x}{y} = xy^{-1}$, therefore $d\cdot\frac{x}{y} =$ (Art. 17.) $y^{-1}dx - xy^{-2}dy = \frac{dx}{y} - \frac{xdy}{y^2} = \frac{ydx-xdy}{y^2}$.

Hence we may differentiate a quotient by Rule 5.

The partial fluxions of $\frac{x}{y}$ are $\frac{du}{dx}dx = \frac{dx}{y}$

and $\frac{du}{dy}dy = \frac{-xdy}{y^2}$.

Cor. $d\cdot\frac{a}{y} = \frac{-ady}{y^2}$.

Examples to Rule 5.

$u = x^2y^3 \therefore du = y^3 \times 2xdx + x^2 \times 3y^2dy.$

$u = x^2 \times (a^4+y^4)^{\frac{3}{2}} \therefore du = (a^4+y^4)^{\frac{3}{2}} \times 2xdx +$
$x^2 \times (a^4+y^4)^{\frac{1}{2}} \times 6y^3dy.$

$u = \sqrt{a^2+x^2} \times \sqrt{b^2+y^2} = (a^2+x^2)^{\frac{1}{2}} \times (b^2+y^2)^{\frac{1}{2}}$

$\therefore du = \sqrt{b^2+y^2} \times \frac{xdx}{\sqrt{a^2+x^2}} + \sqrt{a^2+x^2}\frac{ydy}{\sqrt{b^2+y^2}}.$

$u = \frac{x}{1+x} \therefore du = \frac{dx}{1+x} - \frac{x \times dx}{(1+x)^2} = \frac{dx}{(1+x)^2}.$

$u = \frac{x^n}{(1+x)^n} = \left(\frac{x}{1+x}\right)^n \therefore du = n\cdot\left(\frac{x}{1+x}\right)^{n-1} \times d\left(\frac{x}{1+x}\right)$

$$= n\left(\frac{x}{1+x}\right)^{n-1} \times \frac{dx}{(1+x)^2} = \frac{nx^{n-1}dx}{(1+x)^{n+1}}.$$

$$u = \frac{x}{x+y} \therefore du = \frac{ydx - xdy}{(x+y)^2}.$$

$$u = \frac{a+x}{b+x} \therefore du = \frac{(b-a)dx}{(b+x)^2}.$$

$$u = \frac{1+x^2}{1-x^2} \therefore du = \frac{4xdx}{(1-x^2)^2}.$$

$$u = \frac{x+y}{z^3} \therefore du = \frac{dx+dy}{z^3} - \frac{(x+y)3dz}{z^4}$$

$$= \frac{z\cdot(dx+dy)-(x+y)3dz}{z^4}.$$

$u = \dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}} =$, multiplying the numerator and denominator by the numerator, $\dfrac{2+2\sqrt{1-x^2}}{2x} = \dfrac{1+(1-x^2)^{\frac{1}{2}}}{x}$

$$\therefore du = \frac{1}{x} \times \frac{-xdx}{\sqrt{1-x^2}} - [1+(1-x^2)^{\frac{1}{2}}]\frac{dx}{x^2}$$

$$= \frac{-dx}{\sqrt{1-x^2}} - \frac{(1+\sqrt{1-x^2})dx}{x^2} = -\frac{dx(1+\sqrt{1-x^2})}{x^2\sqrt{1-x^2}}.$$

$$u = x\cdot(a^2+x^2)\sqrt{a^2-x^2} \therefore du = (a^2+x^2)\sqrt{a^2-x^2}dx$$

$+\sqrt{a^2-x^2}\cdot 2x^2dx - \dfrac{(a^2+x^2)x^2dx}{\sqrt{a^2-x^2}} =$, by reduction,

$$\frac{(a^4+a^2x^2-4x^4)dx}{\sqrt{a^2-x^2}}.$$

Logarithms.

19. *Def.* 1. If $y = a^x$, x is the logarithm of y in a system whose base is a.

It is obvious that in different systems the same number must have different logarithms. In the common system the base is 10; hence the common logarithms of 1, 10, 100, 1000, &c. are 0, 1, 2, 3, &c.; the logarithm of a number between 1 and 10 lies between 0 and 1, between 10 and 100 lies between 1 and 2, &c. &c.

Def. 2. The *modulus* of a system is that number which is the logarithm of 2.71828 . . . in that system.

Hence if M is the modulus of a system whose base is a, $2.71828\ldots = a^{M}$.

Cor. 1. In every system the logarithm of unity is equal to nothing.

Cor. 2. If any quantities be taken in geometrical progression, a^{x}, a^{2x}, a^{3x}, &c. their logarithms x, $2x$, $3x$, &c. are in arithmetical progression.

20. If y be taken successively equal to the numbers 1, 2, 3, &c. the corresponding values of x, as deduced from the equation $y = a^{x}$, are the logarithms of 1, 2, 3, &c.: these may be calculated for any known base by methods which will be given in the following chapter, and when registered, they form a table of logarithms.

Briggs', or the common system, where the base equals 10, is the most convenient for arithmetical computation. In the Naperian or hyperbolick system, $a = 2.71828\ldots$ for which e is commonly substituted, these are principally used in analytical calculations, because the modulus is unity. The characteristick for the Naperian most generally adopted is l, the common logarithm is denoted by *log.*, and L is the symbol for the logarithm in a system whose base is a. Thus, since in every system the logarithm of the base is unity, we have $1 = le = \log. 10 = \text{L}a$.

Also conversely, $y = 10^{x}$ if $x = \log. y$, or $= e^{x}$ if $x = ly$, or $= a^{x}$ if $x = \text{L}y$.

The Exponential Theorem.

21. $a^{x} = 1 + \dfrac{\text{A}x}{1} + \dfrac{\text{A}^2x^2}{1.2} + \dfrac{\text{A}^3x^3}{1.2.3} + \dfrac{\text{A}^4x^4}{1.2.3.4} + \&c.$

For $a^{x} = (1 + a - 1)^{x} = (1 + b)^{x}$ if $b = a - 1$

$$= 1 + xb + x.\frac{x-1}{2}b^2 + x\,\frac{x-1}{2}\cdot\frac{x-2}{3}b^3 + \&c.$$

which, arranged according to the powers of x

$$= 1 + (b - \tfrac{1}{2}b^2 + \tfrac{1}{3}b^3 - \&c.)x + px^2 + qx^3 + rx^4 + \&c.$$

where p, q, r, &c. are independent of x,

$$= 1 + \text{A}x + px^2 + qx^3 + rx^4, \&c.,$$

if $\text{A} = a - 1 - \tfrac{1}{2}(a-1)^2 + \&c.$

Hence $a^{z} = 1 + \text{A}z + pz^2 + qz^4 + \&c.$

$$
\begin{aligned}
\therefore a^x \cdot a^z = 1 &+ \text{A}x + px^2 + qx^3 + \&c. \\
&+ \text{A}z + \text{A}^2xz + \text{A}px^2z + \&c. \\
&\qquad + pz^2 + \text{A}pxz^2 + \&c. \\
&\qquad\qquad + qz^3 + \&c. \\
&\qquad\qquad\qquad + \&c.
\end{aligned}
$$

But $a^x \cdot a^z = a^{x+z} = 1 + \text{A}(x+z) + p(x+z)^2 + q(x+z)^3 + \&c.$ which two series being identical, the coefficients of corresponding terms are equal (Alg. 346.); hence $px^2 + \text{A}^2xz + pz^2 = p(x+z)^2$, $qx^3 + \text{A}px^2z + \text{A}pxz^2 + qz^3 = q(x+z)^3$, &c. = &c.; therefore $\text{A}^2 = 2p$, $\text{A}p = 3q$, &c. = &c.; hence

$$p = \frac{\text{A}^2}{2},\ q = \frac{\text{A}^3}{2.3},\ \&c. = \&c., \text{ and}$$

$$a^x = 1 + \frac{\text{A}x}{1} + \frac{\text{A}^2x^2}{1.2} + \frac{\text{A}^3x^3}{1.2.3} + \&c.$$

Cor. Since $(a^x)^n = a^{nx}$, therefore

$$\left\{1 + \frac{\text{A}x}{1} + \frac{\text{A}^2x^2}{1.2} + \frac{\text{A}^3x^3}{1.2.3} + \&c.\right\}^n = 1 + \frac{n\text{A}x}{1} + \frac{n^2\text{A}^2x^2}{1.2} + \frac{n^3\text{A}^3x^3}{1.2.3} + \&c.$$

22. *The quantity* A *in the above series is equal to the hyperbolick logarithm of the base.*

For suppose $x = 1$, then $a = 1 + \frac{\text{A}}{1} + \frac{\text{A}^2}{1.2} + \frac{\text{A}^3}{1.2.3} + \&c.$ but we wish to obtain A in terms of a; suppose then $\text{A} = 1$, and we have $a = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \&c. = 2.71828... = e$, *i.e.* when $a = e$, A is $= 1$, hence $e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$; and, consequently, $e^{\text{A}} = 1 + \frac{\text{A}}{1} + \frac{\text{A}^2}{1.2} + \&c. = a$, or $\text{A} = la$.

Cor. 1. The modulus (M) is equal to unity divided by the hyperbolick logarithm of the base.

For since $a^x = 1 + \frac{\text{A}x}{1} + \frac{\text{A}^2x^2}{1.2} + \&c.$ therefore $a^{\frac{1}{\text{A}}} = 1 + \frac{1}{1} + \frac{1}{1.2} + \&c. = e$, and, consequently, $\frac{1}{\text{A}} = \text{M}$ *.

---

* Lagrange and other writers make A the modulus of the system, or they define the modulus to be that number, the inverse of which is the logarithm of 2.71828....

Cor. 2. $A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 + \&c.$; but this series does not converge, and to express A in a converging series, we have $a = e^A$, therefore $\frac{1}{a} = e^{-A}$, which shows, that if we change A into $-A$, a becomes $\frac{1}{a}$, which substituted, gives

$$-A = \left(\frac{1}{a} - 1\right) - \frac{1}{2}\left(\frac{1}{a} - 1\right)^2 + \frac{1}{3}\left(\frac{1}{a} - 1\right)^3 - \&c.$$

or $A = \frac{a-1}{a} + \frac{1}{2}\cdot\left(\frac{a-1}{a}\right)^2 + \frac{1}{3}\cdot\left(\frac{a-1}{a}\right)^3 + \&c.$ a converging series.

Hence $l.10 = \cdot 9 + \frac{1}{2} \times (\cdot 9)^2 + \frac{1}{3} \times (\cdot 9)^3 + \&c.$
$= 2\cdot 302585093$, &c.

23. *The logarithm of the product of two numbers is equal to the sum of their logarithms, and the logarithm of the quotient is equal to the difference of their logarithms.*

Let $y = a^x$ and $y' = a^{x'}$, therefore $y \cdot y' = a^{x+x'}$ and $\frac{y}{y'} = a^{x-x'}$, *i. e.* $L.yy' = x + x' = Ly + Ly'$;
and $L.\frac{y}{y'} = x - x' = Ly - Ly'$.

24. *The logarithm of a number raised to the* n*th power is equal to* n *times the logarithm of the number.*

For since $y = a^x$, $y^n = a^{nx}$ or $L.y^n = nx = n.Ly$.

25. *The logarithms of the same number in different systems are to each other as the moduli of the systems*.*

For let x and y be the logarithms of the same number z in systems whose bases are a and b; then $a^x = z = b^y$, therefore $xla = ylb$ and $x : y :: \frac{1}{la} : \frac{1}{lb}$, which is the ratio of the moduli of the systems.

Cor. Hence $Ly = M \times ly$, or $ly = A.Ly$.

From this equation, given the logarithm of a number in any known system, we can calculate its logarithm in any other system.

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* Since 0, 1, 2, 3, &c. are the measures of the ratios which a^0, a^1, a^2, a^3, &c. bear to unity, Cotes considers logarithms as the *measures of ratios:* the *modular* ratio is that ratio whose measure is the modulus of the system; and this by Art. 19, Def. 2, is the same in every system, viz. 2.71828.

In Briggs's system M = .434294482, hence log. y = .434294482 ... × ly, or ly = 2.302585093... × log. y, and l.10 = 2.302585093...

26. *To differentiate an exponential of one variable.*

Let $u=a^x$, then

$$\mathrm{U}=a^{x+h}=a^x\times a^h, \text{ which may be shown as in art. 21. to}$$
$$=a^x\{1+\mathrm{A}h+ph^2+qh^3+\&c.\}, \text{ where}$$
$$\mathrm{A}=a-1-\tfrac{1}{2}(a-1)^2+\&c.$$

hence $\frac{\mathrm{U}-u}{h}=a^x\{\mathrm{A}+ph+qh^2+\&c.\}$; and diminishing h without limit, we have $\frac{du}{dx}=\mathrm{A}a^x$, or $du=\mathrm{A}.a^x.dx$.

Otherwise. $u=a^x=$(Theor. 21.)$1+\frac{\mathrm{A}x}{1}+\frac{\mathrm{A}^2x^2}{1.2}+\frac{\mathrm{A}^3x^3}{1.2.3}+\&c.$;

therefore $du=\mathrm{A}dx+\frac{\mathrm{A}^2xdx}{1}+\frac{\mathrm{A}^3x^2dz}{1.2}+\&c.$

$$=\mathrm{A}dx\{1+\frac{\mathrm{A}x}{1}+\frac{\mathrm{A}^2x^2}{1.2}+\&c.\}=\mathrm{A}a^xdx.$$

Hence, RULE 7. *Multiply the exponential by the hyperbolick logarithm of the base, and also by the fluxion of the exponent.*

27. *To differentiate a logarithm.*

Let $u=\mathrm{L}x$: we cannot in this case, as in art. 21, expand u in a series of the form $\mathrm{A}+\mathrm{B}x+\mathrm{C}x^2+\&c.$, for when $x=0$, $\mathrm{L}x=\infty$; nor can we expand it in a descending series of the form $\mathrm{A}+\frac{\mathrm{B}}{x}+\frac{\mathrm{C}}{x^2}+\&c.$, for when $x=1$, $\mathrm{L}x=0$; but since $u=\mathrm{L}x$, $a^u=x$; let $\mathrm{U}-u=k$, then

$$h=a^{\mathrm{U}}-a^u=a^u\{a^k-1\}$$
$$=a^u\{(1+b)^k-1\}, \text{ if } a=1+b$$
$$=a^u\left\{(b-\frac{b^2}{2}+\frac{b^3}{3}-\&c.)k+\mathrm{B}k^2+\mathrm{C}k^3+\&c.\right\} \text{ by (21.)}$$
$$=a^u\{\mathrm{A}k+\mathrm{B}k^2+\mathrm{C}k^3+\&c.\};$$

therefore $1=a^u.\frac{k}{h}.\left\{\mathrm{A}+\mathrm{B}k+\mathrm{C}k^2+\&c.\right\}$, which equation obtains whatever be the values of h and k; but if we take $\frac{k}{h}$ in the limit, it equals $\frac{du}{dx}$, hence $1=a^u.\frac{du}{dx}.\mathrm{A}$, there-

fore $du = \frac{1}{A} \cdot \frac{dx}{a^u} = M \cdot \frac{dx}{x}$.

Otherwise. Since $x = a^u$, differentiating by the preceding article, we have $dx = A.a^u du$, therefore $du = M.\frac{dx}{x}$.

Cor. In the Naperian system $M = 1$; hence if $u = lx$, $du = \frac{dx}{x}$.

Hence, RULE 8. *To differentiate the hyperbolick logarithm of any quantity, divide the fluxion of the quantity by the quantity itself.*

28. *To differentiate an exponential of two variables.*

Let $u = y^x$, $U = (y + k)^{x+h} = (y + k)^x \times (y + k)^h$, by expanding the one as an exponential and the other as a binomial =

$$\left\{1 + l(y + k)h + l(y + k)^2 \cdot \frac{h^2}{1.2} + \&c.\right\} \times$$

$$\left\{y^x + xy^{x-1}k + x \cdot \frac{x-1}{2} \cdot y^{x-2}k^2 + \&c.\right\};$$

therefore $U - u = xy^{x-1}k + x \cdot \frac{x-1}{2} \cdot y^{x-2}k^2 + \&c. +$

$$\left\{l(y + k)h + l(y + k)^2 \cdot \frac{h^2}{1.2} + \&c.\right\}(y + k)^x;$$

therefore $\frac{U-u}{h} = xy^{x-1}\frac{k}{h} + x \cdot \frac{x-1}{2} \cdot y^{x-2}\frac{k^2}{h} + \&c. +$

$$\left\{l(y + k) + l(y + k)^2 \cdot \frac{h}{1.2} + \&c.\right\}(y + k)^x;$$

and taking the increments in their limit,

$\frac{du}{dx} = xy^{x-1}\frac{dy}{dx} + ly \times y^x$, or $du = xy^{x-1}dy + ly \times y^x dx$.

Otherwise. Since $u = y^x$, $lu = xly$ (24.), and differentiating (27, Rule 8.),

$\frac{du}{u} = \frac{xdy}{y} + lydx$, and $du = xy^{x-1}dy + ly\, y^x dx$.

29. The partial fluxions of u are $\frac{du}{dx}dx = ly\, y^x dx$,

(26, Rule 7.), and $\frac{du}{dy}dy = xy^{x-1}dy$ (14, Rule 4.), so that

it is true also in this case, that the total fluxion of u is equal to the sum of its partial fluxions.

Cor. 1. The fluxion of

$y^{xz} = xzy^{xz-1}dy + ly \times y^{xz}\{xdz + zdx\}$.

Cor. 2. The fluxion of

$y^{x^z} = x^z . y^{x^z-1}dy + ly \times y^{x^z}\{zx^{z-1}dx + lx\, x^z dz\}$.

The partial fluxions of y^{xz} are

$$\frac{du}{dx}dx = ly\, y^{xz}zdx = ly \times uzdx,$$

$$\frac{du}{dy}dy = xzy^{xz-1}dy = \frac{uxz}{y}\, . dy,$$

$$\frac{du}{dz}dz = ly\, y^{xz}xdz = lyuxdz.$$

The partial fluxions of y^{x^z} are

$$\frac{du}{dx}dx = lyuzx^{z-1}dx,$$

$$\frac{du}{dy}dy = \frac{ux^z}{y}dy,$$

$$\frac{du}{dz}dz = lx \times lyux^z dz.$$

30. *Examples to Rule* 8.

1. $u = l.\dfrac{1-x^{\frac{3}{2}}}{1+x^{\frac{3}{2}}}$

$$\therefore (23) u = l.(1-x^{\frac{3}{2}}) - l.(1+x^{\frac{3}{2}}) \therefore du = -\tfrac{3}{2}.\frac{x^{\frac{1}{2}}dx}{1-x^{\frac{3}{2}}} - \tfrac{3}{2}.\frac{x^{\frac{1}{2}}dx}{1+x^{\frac{3}{2}}}$$

$$= \frac{-3x^{\frac{1}{2}}dx}{1-x^3}.$$

2. $u = al.\dfrac{b + \sqrt{b^2-y^2}}{y} - \sqrt{b^2-y^2} \therefore du = \dfrac{(y^2-ab)dy}{y\sqrt{b^2-y^2}}$.

3. $u = l.x^n$

$$\therefore du = \frac{nx^{n-1}dx}{x^n} = \frac{ndx}{x}.$$

Otherwise. $u = nl.x \therefore du = \dfrac{ndx}{x}$.

4. $u = (l.x^n)^m \therefore du = m.(l.x^n)^{m-1} \times d.l.x^n$

$$= mn(l.x^n)^{m-1}\frac{dx}{x}.$$

5. $u = l.\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}.$

Multiplying the numerator and the denominator by the numerator,

$$u = l.\frac{2a+2\sqrt{a^2-x^2}}{2x} = l.(a+\sqrt{a^2-x^2}) - lx$$

$$\therefore du = \frac{-xdx}{\sqrt{a^2-x^2}.(a+\sqrt{a^2-x^2})} - \frac{dx}{x} = \frac{-adx}{\sqrt{a^2-x^2}}.$$

6. $u = l^2x$, where l^2 is the same as $l.l.$

Substitute $y = lx \therefore u = ly \therefore du = \dfrac{dy}{y} = \dfrac{dx}{xlx}.$

7. $u = \dfrac{1}{\sqrt{-1}}.l(x\sqrt{-1}+\sqrt{1-x^2}).$

$$\therefore du = \frac{1}{\sqrt{-1}}.\frac{\left(dx\sqrt{-1}-\dfrac{xdx}{\sqrt{1-x^2}}\right)}{x\sqrt{-1}+\sqrt{1-x^2}}$$

$$= \frac{dx}{\sqrt{1-x^2}}.\frac{(\sqrt{-1}\sqrt{1-x^2}-x)}{-x+\sqrt{-1}\sqrt{1-x^2}} = \frac{dx}{\sqrt{1-x^2}}.$$

31. *To differentiate circular functions.*

(1.) $u = \sin.x.$

$\upsilon = \sin.x(x+h)$ and

$$\upsilon - u = \sin.(x+h) - \sin.x$$
$$= (\text{Trig. p. 27.})\sin.x\cos.h + \cos.x\sin.h - \sin.x$$
$$= \cos.x\sin.h - (1-\cos.h)\sin.x$$
$$= \cos.x\sin.h - \text{vs}.h\sin.x;$$

therefore $\dfrac{\upsilon - u}{h} = \dfrac{\cos.x\sin.h - \text{vs}.h\sin.x}{h}$; and to find the value of this in the limit, diminish h indefinitely, and vs.h becomes indefinitely small when compared with sin.h; also sin.h : chord$h$: tan.h : : 1 : 1 : 1 in the limit; for the opposite angles are supplements to each other, and (Trig. p. 27), the sides are as the sines of their opposite angles; hence *à fortiori* h : sin.h : : 1 : 1, and we have

$$\frac{du}{dx} = \frac{\cos.x.h}{h} = \cos.x, \text{ and } du = \cos.xdx.$$

(2.) $u = \cos.x$.

$$\text{U} - u = \cos.(x+h) - \cos.x$$
$$= (\text{Trig. p. 28}),\ \cos.x\cos.h - \sin.x\sin.h - \cos.x$$
$$= -\sin.x\sin.h - \cos.x(1-\cos.h)$$
$$= -\sin.x\sin.h - \cos.x\ \text{vs}\cdot h;$$

therefore $\dfrac{\text{U}-u}{h} = \dfrac{-\sin.x\sin.h - \cos.x\ \text{vs}.h}{h}$ and

$$\frac{du}{dx} = \frac{-\sin.x.h}{h} = -\sin.x, \text{ or } du = -\sin.xdx.$$

(3.) $u = \text{vs}.\ x$.

$$\text{U} - u = \text{vs}.(x+h) - \text{vs}.x = 1 - \cos.(x+h) - (1-\cos.x)$$
$$= \cos.x - (\cos.x\cos.h - \sin.x\sin.h)$$
$$= \cos.x(1-\cos.h) + \sin.x\sin.h$$
$$= \cos.x\ \text{vs}.h + \sin.x\sin.h;$$

therefore $\dfrac{\text{U}-u}{h} = \dfrac{\cos.x\cdot\text{vs}.h + \sin.x\sin.h}{h}$ and

$$\frac{du}{dx} = \frac{\sin.x.h}{h} = \sin.x \text{ and } du = \sin.xdx.$$

(4.) $u = \tan.x$.

$$\text{U} - u = \tan.(x+h) - \tan.x$$
$$= (\text{Trig. p. 35})\ \frac{\tan.x + \tan.h}{1 - \tan.x\tan.h} - \tan.x$$
$$= \frac{\tan.h + \tan.^2x\tan.h}{1-\tan.x\tan.h} = \frac{(1+\tan.^2x)\tan.h}{1-\tan.x.\tan.h}$$
$$= \frac{\sec.^2x\tan.h}{1-\tan x.\tan.h},$$

and $\tan.h : h$ in the limit $:: 1 : 1$, therefore $\dfrac{du}{dx} = \sec.^2x$, or $du = \sec.^2xdx$.

(5.) $u = \cot.x$.

To deduce this from the preceding, we have

$\cot.x = \tan.\left(\dfrac{\pi}{2} - x\right)$, if $\dfrac{\pi}{2}$ is a quadrant, therefore

$$du = -\sec.^2\left(\frac{\pi}{2} - x\right)dx = -\text{cosec}^2xdx.$$

(6.) $u = \sec.x$.

$$\text{U} - u = \sec.(x+h) - \sec.x$$
$$= \frac{1}{\cos.(x+h)} - \frac{1}{\cos.x} = \frac{\cos.x - (\cos.x\cos.h - \sin.x\sin.h)}{\cos.x.\cos.(x+h)}$$

$$=\frac{\sin x \sin h+\cos x(1-\cos h)}{\cos x \cos(x+h)}=\frac{\sin x \sin h+\cos x \,\mathrm{vs} x}{\cos x \cos(x+h)};$$

therefore $\frac{du}{dx}=\frac{\sin.x}{\cos.^2 x}=\tan.x\sec.x$ and $du=\tan x\sec.xdx$.

(7.) $u = \text{cosec}.x$

$$u=\sec.\left(\frac{\pi}{2}-x\right);\text{ therefore } du=-\text{cotan}.x\ \text{cosec}.xdx.$$

Hence collecting the formulæ,

(1.) $d.\sin.x = dx\cos.x$,

(2.) $d.\cos.x = -dx\sin.x$.

(3.) $d.\text{vs}.\ x = dx\sin.x$.

(4.) $d.\tan.x = dx\sec.^2 x$.

(5.) $d.\text{cotan}x = -dx\,\text{cosec}.^2 x$.

(6.) $d.\sec.x = \frac{dx\sin.x}{\cos.^2 x} = dx\tan.x\sec.x$.

(7.) $d.\text{cosec}x = -dx\frac{\cos.x}{\sin.^2 x} = -dx\cot.x\,\text{cosec}.x$.

32. In the preceding article, the sine, cosine, and the other trigonometrical lines are considered as functions of the arc to which they belong, and the fluxion of the function is found in each case in terms of the fluxion of the principal variable and of a trigonometrical line which, as will be shown in Chap. 4, can be expanded in a series of that variable; but it is sometimes convenient that the same formulæ should be put under a different form. Instead of considering the arc as the principal variable, and the sine or cosine as the function, let us take the inverse of this, and suppose the arc to be the function depending upon the value of its sine or cosine.

Mr. Herschell has proposed a very convenient notation for inverse functions. If $u = \mathrm{F}x$, then the function that x is of u is expressed by $x = \mathrm{F}^{-1}u$. Hence according to this notation, if u is the arc, and x its sine, *i. e.* if $x = \sin.u$, $u = \sin.^{-1}x$: $u = \tan.^{-1}x$ shows that x is the tangent of u, or that u = arc, whose tangent $= x$. Similarly $u = l^{-1}x$, shows that x is the hyperbolick logarithm of u.

(1.) Let $u=\text{arc}\sin.=x$, or $u=\sin.^{-1}x$.

By (1.) of preceding article,

$$dx=du\cos.u \therefore du=\frac{dx}{\cos.u}=\frac{dx}{\sqrt{1-x^2}}.$$

(2.) $u = \cos.^{-1}x$.

By (2.) of art. 31,

$$dx = -du \sin.u \therefore du = \frac{-dx}{\sin.u} = \frac{-dx}{\sqrt{1-x^2}}.$$

(3.) $u = \text{vs.}^{-1}x$.

By (3.) $dx = du \sin.u \therefore du = \dfrac{dx}{\sin.u} = \dfrac{dx}{\sqrt{2x - x^2}}$.

(4.) $u = \tan.^{-1}x$.

By (4.) $dx = du \sec.^2 u \therefore du = \dfrac{dx}{\sec.^2 u} = \dfrac{dx}{1+x^2}$.

(5.) $u = \text{cotan.}^{-1}x$.

By (5.) $dx = -du \,\text{cosec.}^2 u \therefore du = \dfrac{-dx}{\text{cosec.}^2 u} = \dfrac{-dx}{1+x^2}$.

(6.) $u = \sec.^{-1}x$.

By (6.) $dx = du \tan.u \sec.u \therefore du = \dfrac{dx}{\tan.u \sec.u} = \dfrac{dx}{x\sqrt{x^2-1}}$.

(7.) $u = \text{cosec.}^{-1}x$.

By (7.) $dx = -du \tan.u \sec.x \therefore du = \dfrac{-dx}{\tan.u \sec.u}$

$$= \frac{-dx}{x\sqrt{x^2-1}}.$$

Collecting these formulæ and reducing them to radius $= a$ (Trig. p. 19), we have

$$(1.)\ d\sin.^{-1}x = \frac{adx}{\sqrt{a^2-x^2}}.$$

$$(2.)\ d.\cos.^{-1}x = \frac{-adx}{\sqrt{a^2-x^2}}.$$

$$(3.)\ d.\text{vs.}^{-1}x = \frac{adx}{\sqrt{2ax-x^2}}.$$

$$(4.)\ d.\tan.^{-1}x = \frac{a^2dx}{a^2+x^2}.$$

$$(5.)\ d.\text{cotan.}^{-1}x = \frac{-a^2dx}{a^2+x^2}.$$

$$(6.)\ d.\sec.^{-1}x = \frac{a^2dx}{x\sqrt{x^2-a^2}}.$$

$$(7.)\ d.\text{cosec.}^{-1}x = \frac{-a^2dx}{x\sqrt{x^2-a^2}}.$$

The fluxions of u in (2), (5) and (7) are negative, because as the arc increases, the trigonometrical lines decrease.

33. Thus, from the definition in Art. 7, rules have been established for the differentiation of any function whatever, algebraical or transcendental; and it has been shown generally that if $u = \text{F}x$, $du = \phi x dx$ where ϕx depends solely upon the form of the function $\text{F}x$, and is wholly independent of the increment we assign to x.

Since $du = \phi x dx$, we have $\dfrac{du}{dx} = \phi x$, which, if it contain x, is a variable quantity and may be again differentiated, and we have $d \cdot \dfrac{du}{dx} = d \cdot \phi x = \psi x dx$, and dividing by dx,

$$\frac{d \cdot \frac{du}{dx}}{dx} = \psi x.$$ The operation may be continued unless x ceases to enter into the function.

34. If we consider u as consisting of different functions, it is manifest that the same rules must obtain; for since they are true for every particular form of each function, they must be universally true. As an example to illustrate this position, suppose $v = \text{F}x$ and $w = fy$, and let it be required to show that $d.(vw) = wdv + vdw$: take particular functions, suppose $v = x^m$ and $w = ly$, then $vw = x^m \times ly$ and $d(vw)$ $= d.(x^m ly) = ly \times mx^{m-1}dx + x^m . \dfrac{dy}{y} = wdv + vdw$; and thus the rule for differentiating a product may be shown to obtain in any proposed case. We shall, however, give general demonstrations of these rules in the 3d Chapter, founded upon a Theorem which is due to Lagrange.

Examples.

Ex. 1. $u = \text{arc. sin.} = 2y\sqrt{1-y^2}$; or $u = \sin.^{-1}. 2y\sqrt{1-y^2}$.

Substitute $x = 2y\sqrt{1-y^2}$ $\therefore du = \dfrac{dx}{\sqrt{1-x^2}}$; but $dx = dy$ $\left\{2\sqrt{1-y^2} - \dfrac{2y^2}{\sqrt{1-y^2}}\right\} = \dfrac{2dy.}{\sqrt{1-y^2}}(1-2y^2)$. Also $1-x^2$ $= 1 - 4y^2 + 4y^4$ $\therefore \sqrt{1-x^2} = 1-2y^2$, hence $du = \dfrac{2dy}{\sqrt{1-y^2}}$.

Ex. 2. $u = \tan.^{-1}. \dfrac{x-y}{x+y}$.

Substitute $t = \dfrac{x-y}{x+y}$ $\therefore$ (32.4) $du = \dfrac{dt}{1+t^2}$; but

$$dt = \frac{dx-dy}{x+y} - \frac{(x-y)(dx+dy)}{(x+y)^2} = \frac{-2xdy+2ydx}{(x+y)^2}\text{; and}$$

$$1+t^2 = 1+\left(\frac{x-y}{x+y}\right)^2 = \frac{2x^2+2y^2}{(x+y)^2} \therefore du = \frac{ydx-xdy}{x^2+y^2}.$$

Ex. 3. $u = \text{arc. tan.} \dfrac{u}{2} = x$.

Let $t = \tan. u \therefore$ (Trig. p. 62) $t = \dfrac{2x}{1-x^2} \therefore dt = \dfrac{2dx}{1-x^2} + \dfrac{4x^2dx}{(1-x^2)^2}$

$= \dfrac{(1+x^2)2dx}{(1-x^2)^2}$, and $1+t^2 = 1+\dfrac{4x^2}{(1-x^2)^2} = \dfrac{(1+x^2)^2}{(1-x^2)^2} \therefore$

$du = \dfrac{2dx}{1+x^2}$.

Ex. 4. $u = x. \sin.^{-1}x$

$$\therefore du = dx. \sin.^{-1}x + \frac{xdx}{\sqrt{1-x^2}}.$$

Ex. 5. $u = (\sin.^{-1}x)^2$

$$\therefore du = 2. \sin.^{-1}x. \frac{dx}{\sqrt{1-x^2}}.$$

35. *Additional Examples to all the Rules.*

$$u = \frac{y}{b-y} \therefore du = \frac{bdy}{(b-y)^2}.$$

$$u = \left(\frac{x}{a+x}\right)^2 \therefore du = \frac{2axdx}{(a+x)^3}.$$

$$u = \frac{3xy}{a-x} \therefore du = 3. \frac{(axdy+aydx-x^2dy)}{(a-x)^2}.$$

$$u = \frac{\sqrt{1+x}}{\sqrt{1-x}} \therefore du = \frac{dx}{(1+x)^{\frac{1}{2}}.(1-x)^{\frac{3}{2}}}.$$

$$u = \frac{2}{n}\left\{\sqrt{1+x^n} + \frac{1}{\sqrt{1+x^n}}\right\} \therefore du = \frac{x^{2n-1}dx}{(1+x^n)^{\frac{3}{2}}}.$$

$$u = \frac{x^2+2y^2}{y^{\frac{1}{2}}} \therefore du = \frac{2xydx+3y^2dy-\frac{1}{2}x^2dy}{y^{\frac{3}{2}}}.$$

$$u = \frac{x^{\frac{1}{2}}-(x-2)^{\frac{1}{2}}}{x^{\frac{1}{2}}+(x-2)^{\frac{1}{2}}} \therefore du = -\frac{(x^{\frac{1}{2}}-(x-2)^{\frac{1}{2}})^2.dx}{2\sqrt{x^2-2x}}.$$

$$u = \frac{x-\sqrt{x+1}}{x+\sqrt{x+1}} \therefore du = \frac{(x+2)dx}{\sqrt{x+1}.(x+\sqrt{x+1})^2}.$$

$$u = \frac{x\sqrt{1+x^2}}{\sqrt{1-x^2}} \therefore du = \frac{1+2x^2-x^4}{\sqrt{1+x^2}.(1-x^2)^{\frac{3}{2}}}.$$

$$u = \sqrt{ax+x^2+\sqrt[4]{a^4-x^4}} \therefore$$

$$du = \frac{1}{2}dx\frac{\{(a+2x)(a^4-x^4)^{\frac{3}{4}}-x^3\}}{(ax+x^2+\sqrt[4]{a^4-x^4})^{\frac{1}{2}}.(a^4-x^4)^{\frac{3}{4}}}.$$

$$u = \frac{a^2+x^2}{\sqrt{ax+x^2}} \therefore du = \frac{1}{2}dx.\frac{\{3ax^2+2x^3-a^3-2a^2x\}}{(ax+x^2)^{\frac{3}{2}}}$$

$$u = \frac{x-\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}} \therefore$$

$$du = \frac{2(x-\sqrt{x^2-y^2})^2}{y^3\sqrt{x^2-y^2}}.(xdy-ydx).$$

$$u = \frac{x\sqrt{ax+x^2}}{a\sqrt{ay-xy}} \therefore$$

$$du = \frac{3a^2yxdx+2ayx^2dx-3yx^3dx-a^2x^2dy+x^4dy}{2a\sqrt{ax+x^2}\times(ay-xy)^{\frac{3}{2}}}.$$

$$u = \frac{a^2-x^2}{a^4+a^2x^2+x^4} \therefore du = -2x\frac{(2a^4+2a^2x^2-x^4)dx}{(a^4+a^2x^2+x^4)^2}.$$

$$u = l.\frac{x}{\sqrt{a^2+x^2}} \therefore du = \frac{a^2dx}{x.(a^2+x^2)}.$$

$$u = a^{x^x} \therefore du = la \times x^x a^{x^x}\{lx+1\}dx.$$

$$u = l.xe^{\cos.x} \therefore du = \frac{dx}{x}(1-x\sin.x).$$

$$u = lx.e^{\cos.x} \therefore du = e^{\cos.x}dx\left\{\frac{1}{x}-lx\sin.x\right\}.$$

$$u = l.(x + \sqrt{1+x^2}) \therefore du = \frac{dx}{\sqrt{1+x^2}}.$$

$$u = l.\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right)^{\frac{1}{2}} \therefore du = \frac{dx}{\sqrt{1+x^2}}.$$

$$u = \sin.^{-1}.\frac{1-x^2}{1+x^2} \therefore du = \frac{-2dx}{1+x^2}.$$

$$u = \tan.^{-1}.\sqrt{\frac{1-x}{1+x}} \therefore du = \frac{-\frac{1}{2}dx}{\sqrt{1-x^2}}.$$

$$u = \sin.^{-1}.\frac{x}{\sqrt{1+x^2}} = \tfrac{1}{2}\sin.^{-1}.\frac{2x}{1+x^2} = \cos.^{-1}.\frac{1}{\sqrt{1+x^2}}$$

$$= \tfrac{1}{2}\cos.^{-1}.\frac{1-x^2}{1+x^2} = \cot.^{-1}.\frac{1}{x} = \sec.^{-1}.\sqrt{1+x^2}$$

$$= \tfrac{1}{2}\sec.^{-1}.\frac{1+x^2}{1-x^2} = \operatorname{cosec}.^{-1}.\frac{\sqrt{1+x^2}}{x} = \tfrac{1}{2}\operatorname{cosec}.^{-1}.\frac{1+x^2}{2x}$$

$$= \tfrac{1}{2}\operatorname{vs}.^{-1}.\frac{2x^2}{1+x^2}. \text{ In each case } du = \frac{dx}{1+x^2}.$$

$$u = \tan.^{-1}.x\sqrt{\frac{b}{a}} = \sin.^{-1}.\sqrt{\frac{bx^2}{a+bx^2}} = \tfrac{1}{2}\sin.^{-1}.\frac{2x\sqrt{ab}}{a+bx^2}$$

$$= \cos.^{-1}.\sqrt{\frac{a}{a+bx^2}} = \tfrac{1}{2}\cos.^{-1}.\frac{a-bx^2}{a+bx^2} = \cot.^{-1}.\frac{\sqrt{a}}{x\sqrt{b}}$$

$$= \sec.^{-1}.\sqrt{\frac{a+bx^2}{a}} = \tfrac{1}{2}\sec.^{-1}.\frac{a+bx^2}{a-bx^2}$$

$$= \operatorname{cosec}.^{-1}.\sqrt{\frac{a+bx^2}{bx^2}} = \tfrac{1}{2}\operatorname{cosec}.^{-1}.\frac{a+bx^2}{2x\sqrt{ab}}$$

$$= \tfrac{1}{2}\operatorname{vs}.^{-1}.\frac{2bx^2}{a+bx^2}. \text{ In each case } du = \frac{dx}{a+bx^2}.$$

(Hirsch's Tables.)

$$u = \sqrt{1-x^2} \times \sin.^{-1}.x \therefore du = \frac{-xdx}{\sqrt{1-x^2}}\sin.^{-1}.x + dx.$$

$$u = \frac{x\sin.^{-1}.x}{\sqrt{1-x^2}} + l.\sqrt{1-x^2} \therefore du = \frac{dx\sin.^{-1}x}{(1-x^2)^{\frac{3}{2}}}.$$

$$u=\frac{\sin.^{-1}.x}{\sqrt{1-x^2}}+l.\sqrt{\frac{1-x}{1+x}}\therefore du=\frac{xdx\sin.^{-1}x}{(1-x^2)^{\frac{3}{2}}}.$$

$$u=(\cot.^{-1}.x)^2\therefore du=\frac{-2dx}{1+x^2}\times\cot.^{-1}.x.$$

$$u=l.\frac{a^{\frac{n}{2}}+x^{\frac{n}{2}}}{a^{\frac{n}{2}}-x^{\frac{n}{2}}}\therefore du=\frac{na^{\frac{n}{2}}x^{\frac{n}{2}-1}dx}{a^n-x^n}.$$

$$u=x\times e^{\tan.x}\therefore du=\frac{1+x+x^2}{1+x^2}e.^{\tan.x}dx.$$

$$u=\cos.x^{\sin.x}\therefore du=dx\cos.x^{\sin.x}\left\{\cos.xl\cos.x-\frac{\sin.^2x}{\cos.x}\right\}$$

Garnier's general example is

$$u=\frac{x.a^{\frac{1}{x}}}{l.\sqrt{1-x^2}}+\frac{\sin.x\cos.x}{\sec.^2x}+\frac{\cos.\sqrt{1-x^2}}{\tan.\frac{ax-x^2}{\sqrt{b^2-x^2}}}$$

$$+\tan.^{-1}.(1-\sqrt{1-x^2}).$$

36. In differentiating a complicated analytical expression, it will be found convenient to substitute for parts of it; thus let

$$u=\frac{(a-x)^m.\sqrt[n]{b^2-x^2}}{x^k\sqrt[r]{\frac{a-x}{b+x}}}+\frac{a}{x^2-a^2}-f.\sqrt{ax-x^2}.$$

Substitute $p=\frac{(a-x)^m.\sqrt[n]{b^2-x^2}}{x^k\sqrt[r]{\frac{a-x}{b+x}}}$, $q=\frac{a}{x^2-a^2}$, $r=f.\sqrt{ax-x^2}$

$\therefore du=dp+dq-dr.$

To find dp, substitute $z=(a-x)^m$, $v=\sqrt[n]{b^2-x^2}$, $t=x^k$, $w=\sqrt[r]{\frac{a-x}{b+x}}$, so that $p=\frac{zv}{tw}=\frac{\text{P}}{\text{Q}}$ by substitution $\therefore$

$$dp=\frac{d.\text{P}}{\text{Q}}-\frac{\text{P}d.\text{Q}}{\text{Q}^2}\text{ where }d\text{P}=vdz+zdv$$

$$=-\sqrt[n]{b^2-x^2}.m(a-x)^{m-1}dx-(a-x)^m.\frac{2}{n}(b^2-x^2)^{\frac{1}{n}-1}xdx,$$

and $dQ = \dot{w}dt + tdw =$ (by substitution) $\mathrm{x}^{\frac{1}{r}} . kx^{k-1}dx + \frac{1}{r}x^k \mathrm{x}^{\frac{1}{r}-1} d\mathrm{x} = \mathrm{x}^{\frac{1}{r}-1}\left\{k\mathrm{x}x^{k-1}dx + \frac{1}{r}x^k\left(\frac{-dx}{b+x} - \frac{(a-x)dx}{(b+x)^2}\right)\right\}$

$$= \mathrm{x}^{\frac{1}{r}-1}\left\{k\mathrm{x}x^{k-1}dx - \frac{(a+b)x^k dx}{r.(b+x)^2}\right\}$$

$$\therefore dp = -\frac{(a-x)^{m-1}dx}{x^k \mathrm{x}^{\frac{1}{r}}}\left\{m^n\sqrt{b^2-x^2} - \frac{2(a-x)}{n}x(b^2-x^2)^{\frac{1}{n}-1}\right\}$$

$$\frac{-(a-x)^m . \sqrt[n]{b^2-x^2}}{x^k \mathrm{x}^{\frac{1}{r}-1}}\left\{\frac{k\mathrm{x}}{x} - \frac{a+b}{r.(b+x)^2}\right\}dx,$$

$\mathrm{x} = \frac{a-x}{b+x}$, to which add $d\dot{q}$ and $- df\sqrt{ax - x^2}$ (which last cannot be found unless the form of the function is known) and the result $= du$.

Infinitesimals.

37. The doctrine of infinitesimals is perfectly distinct from that of limiting ratios, and it is by no means necessary to introduce it into a treatise on fluxions; yet, as it frequently enables us to find the limit, and in other respects throws great light on the subject, we shall here endeavour to establish it upon unobjectionable principles.

Mr. Locke, in his "Essay concerning Human Understanding," has shown that we can have no positive idea of an infinite magnitude. "Whether any one has or can have a positive idea of an actual infinite number, I leave him to consider, till his infinite number be so great, that he himself can add no more to it; and as long as he can increase it, I doubt he himself will think the idea he hath of it a little too scanty for positive infinity." Book 2. ch. 17. § 16.

If then we would define infinite magnitude or infinite space, we can only define it from its negative property of being incapable of increase or decrease by the addition or subtraction of any portions of matter or space of which the mind can form an idea.

38. It follows from this definition, that if z represent an infinite and a a finite quantity, $z \pm a = z$; for otherwise z would admit of increase or decrease by the addition or subtraction of a.

The following demonstration has also been given of the same proposition.

Let $\frac{1}{a} + \frac{1}{z} = \text{M}$, then multiplying by az, $a + z = \text{M}az$; suppose z to be increased without limit, then $\frac{1}{z}$ is diminished without limit, and when z is infinite, $\frac{1}{z} = 0$; whence $\frac{1}{a} = \text{M}$, and by substitution $a + z = \frac{\text{M}z}{\text{M}} = z$.

39. *If* a *represent a finite and* x *a variable quantity, which is indefinitely diminished, and at length vanishes,* a + x = a *ultimately.*

This is an axiom, and cannot be demonstrated: but it may be shown to agree with the preceding article.

For take $z : a :: a : x$, then as x decreases, z increases, and if z could be increased so as to become infinite, x would vanish; but comp. $z + a : z = a + x : a$; and $z + a = z$, therefore $a + x = a$ when $z = \infty$ or $x = 0$.

40. *If* y *and* x *are two variable quantities so dependent upon each other that, being gradually diminished, they vanish together, but in a ratio which is infinite, the last ratio of* y + x : y *is a ratio of equality.*

For take $z : a :: y : x$ where a is a finite constant quantity, then comp. $z + a : z :: y + x : y$; but at that point of time at which y and x vanish, y is infinite when compared with x, and, consequently, z is ultimately an infinite quantity, and therefore (38.) $z + a : z$, or its equal $y + x : y$ is a ratio of equality.

Ex. As the arc of a circle decreases, and at length vanishes, its chord and versed sine gradually decrease and vanish at the same point of time, and since diameter : chord :: chord : versed sine, they vanish in an infinite ratio, and, consequently, chord + versed sine : chord is ultimately a ratio of equality.

41. So long as the quantities x and y possess any magnitude, however small, $y + x : y$ is not a ratio of equality; for if x possess magnitude, it can be diminished, and therefore z can be increased, or $z + a$ is not $= z$: they are in this ratio only *when* they vanish *.

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* Whether the variables ever actually attain to this ratio is a question which has been much debated, arising, as it should seem, from the imperfection of language, which, as Mr. Locke observes,

42. The last example proves that indefinitely small quantities may vanish in a finite ratio.

This axiom is the foundation of the whole science, and it cannot be too frequently illustrated; take the ratio $3x + x^2 : 2x + x^2$, and substitute for x any numbers in succession, either whole or fractional, which are in a decreasing series, and it will be seen that the ratio increases, and that it is always less than $3 : 2$; hence its ultimate

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rather 'serves for the upholding common conversation and commerce about the ordinary affairs and conveniences of civil life, in the societies of men one amongst another,' than 'to express, in general propositions, certain and undoubted truths, which the mind may rest upon, and be satisfied with, in its search after true knowledge.' B. 3. c. 9. § 3.

When the limit is obtained by *increasing* the magnitudes indefinitely, the limiting ratio is one to which they can only approximate, and which they can never reach; and the reason is sufficiently obvious, for they can never actually become infinite; but when they are gradually diminished and at length vanish, I can see no objection to the position, that they do actually possess this ratio at the moment of vanishing.

Sir I. Newton has been thought to hold the contrary opinion from the following passage in the Scholium: 'Ultimæ rationes illæ quibuscum quantitates evanescunt, revera non sunt *rationes quantitatum ultimarum,* sed limites ad quos quantitatum sine limite decrescentium rationes semper appropinquant; et quas propius assequi possunt quàm pro datâ quâvis differentiâ, nunquam verò transgredi, neque priùs attingere quàm quantitates diminuuntur in infinitum:' but it is evident from the context, that by '*rationes quantitatum ultimarum,*' he understands the ratios of the quantities while possessing a determinate magnitude; and, in fact, he concludes the Scholium with a similar caution, 'cave intelligas quantitates magnitudine determinatas, sed cogita semper diminuendas sine limite.' Nothing can be more decisive of his opinion than the definition which he gives of the ultimate ratio. 'Per ultimam rationem quantitatum evanescentium, intelligendam esse rationem quantitatum, non antequam evanescunt, non postea, sed *quâcum evanescunt.*'

If, notwithstanding this high authority, the student should be still fearful of committing the solœcism of asserting that magnitudes bear to each other a certain ratio when they vanish, he may avoid the use of the terms *prime and ultimate,* and may define the *limiting ratio* to be one to which the magnitudes gradually approximate, which they approach nearer than by any assignable difference, but which they never reach.

ratio, which is 3 : 2, so far from vanishing, is the greatest which the magnitudes can bear to each other.

43. *In order that the ultimate ratio of evanescent quantities may be a ratio of equality, their difference must not only vanish, but it must become indefinitely small when compared with either of them.*

As an example take $x^2 - a^2$ and $x^2 - ax$, which vanish together when $x = a$, also their difference $ax - a^2$ vanishes at the same time, but since it does not vanish when compared with $x^2 - a^2$ and $x^2 - ax$, the ultimate ratio in this instance is not a ratio of equality: it is $2:1$.

44. *Hence arises the necessity of admitting infinitesimals of different orders.*

For since $1 : x :: x : x^2$ if x be indefinitely diminished, x will be contained in 1 an infinite number of times, and consequently x^2 is contained in x an infinite number of times, or x^2 being indefinitely small compared with x is an infinitesimal of the *second* order.

For the same reason x^3, $x^4 \ldots x^n$ are infinitesimals of the 3d, 4th ... nth orders.

Cor. 1. If an infinitesimal is multiplied by a finite magnitude its order is not changed.

Cor. 2. Hence, if a function be developed in a series ascending by the powers of h, and h be diminished without limit, all the succeeding terms may be neglected when compared with any of the preceding.

45. *There are intermediate orders of infinitesimals.*

For let $y^2 = ax$, then x being taken an infinitesimal of the first order, y which is not finite cannot be of the first order, for then y^2 or ax would be of the second, and is therefore of an intermediate order.

The student should consult the Scholium at the end of the first book of the Principia, where all this is exemplified by the different orders of contact in curves.

46. *The product of two infinitesimals of the first order is an infinitesimal of the second order.*

For let x and y be two infinitesimals of the same order, and take $1 : x :: y : z$, then $z = xy$; but when x is indefinitely small compared with 1, z is indefinitely small compared with y, or is an infinitesimal of the second order.

Similarly it may be shown that the product of three is of the third order, &c. &c.

Cor. Hence, if $u = (x + h)(y + k) = xy + xk + yh + hk$,

and h and k be diminished indefinitely, hk may be neglected, when compared with $xk + yh$.

47. The principle of limiting ratios is in itself free from all difficulty. The first and last ratios of the magnitudes are as precise and definite, and as well adapted in every respect to be the subject of calculation, as any of the intermediate ratios. This principle is frequently represented by foreign writers to be different from that upon which Newton founded his system of Fluxions, and they are fond of considering it as an improvement due to D'Alembert. But that the theories are the same, is sufficiently evident from the second Lemma of the second volume of the Principia, without appealing to the treatise on fluxions, which is a posthumous work. In this Lemma the theory is fully developed; the fluxions are called *Momenta*, because the increments are supposed to be generated by material particles. He repeats the former caution, "Cave tamen intellexeris particulas finitas;" and then adds, with respect to the magnitudes of the momenta generated, "Neque enim spectatur in hoc lemmate magnitudo momentorum; sed *prima nascentium proportio*:" and he then proceeds to show that the momentum of $A^m.B^n = maA^{m-1} + nbB^{n-1}$ where a and b are the momenta of A and B. Newton saw that the ratio of the velocities of the fluents is the same as the limiting ratio of their increments, and adopted the former phraseology as being less liable to metaphysical objections.

It has been objected to Newton's mode of considering the genesis of quantity, that it includes the idea of motion and time, which, it is urged, belong to physical science, and are foreign to the spirit of pure analysis. This would be an unanswerable objection, if in the principle of fluxions, or in its application, we either assumed or even referred to the Laws of Motion, which are founded upon observation and experiment. The science of Analytical Calculation, though it renders great assistance to Natural Philosophy, should accept of none in return *. But the introduction of the simple idea of motion is so far from being an objection, that

* There are elementary writers who have not sufficiently attended to this distinction. I have seen a demonstration, in which it is assumed that the motion of the generating point of a curve is compounded of two motions in the directions of its co-ordinates.

it enters into every system of calculation which has hitherto been proposed. In Arbogast's "Calcul des Derivations" and Lagrange's "Théorie des Fonctions Analytiques," while certain parts of the function are constant, it is supposed that the remaining parts change their state or value, and all change implies motion; so that, with respect to this objection at least, the principle of limiting ratios is as purely algebraical as the theories of these eminent analysts.

The doctrine of Infinitesimals, which was admitted both by Newton and Leibnitz, is not so easily vindicated. The supposition that a quantity may become infinitely smaller than another which has been already assumed to be infinitely small, is obviously absurd or unintelligible. To avoid this inconsistency, we must be careful to consider an infinitesimal, not as representing an absolute quantity, however small, but as representing a ratio which is either infinitely great or infinitely small in the limit, according as we compare it with an infinitesimal of a lower or of a higher order than itself.

Influenced partly by objections such as these, and partly, I fear, by a spirit of jealousy, many writers of the last century looked out for some other principle of calculation than that upon which Sir I. Newton has founded his doctrine of Fluxions. As my object is not only to teach the elements of the science, but also to induce the reader to study the writings of those who have unquestionably greatly enlarged its boundaries, I shall give, in the third Chapter, the principle upon which Lagrange rests his Theory, which the reader will perceive is independent of the doctrine of limits. The great difficulty, however, is to establish the truth of the proposition contained in (44. Cor. 2.) which follows so immediately from the doctrine of infinitesimals. We shall insert Lagrange's demonstration of this proposition in his own words: the student will then be enabled to form an opinion of the relative merit of the two systems.

CHAPTER II.

On Integration.

1. HAVING established in the preceding chapter rules for the differentiation of quantity both algebraick and transcendental, we shall now consider the inverse problem; or the method of finding the fluent of a given fluxion.

If it were required to give a definition of the fluent of a proposed fluxion, we must suppose that *two* fluxions are under consideration, and their fluents may be defined to be two magnitudes, the limiting ratio of whose cotemporary increments is the ratio of the proposed fluxions.

The act of deriving the fluent from its fluxion is called *Integration* *.

In this chapter we chall only investigate the general rules and methods of integration, and apply them to such examples as occur the most frequently; reserving the canonical forms and the remainder of the subject for the second volume.

2. Whatever be the form of the fluent, its fluxion may be found; but the inverse problem cannot be solved in all cases, as the proposed fluxion may be of such a form as cannot arise from the differentiation of any fluent.

3. The symbol adopted to denote the operation of taking the fluent is $\int$.

* I use the terms *Integration* and *Differentiation*, which properly belong to the *Calculus of Differences*, because they are adopted into the language, and I know no other that will answer my purpose. No confusion can arise from this phraseology, as the two sciences of Fluxions and of Increments possess each a peculiar notation and algorithm. Some of our writers, in imitation of the French, even call the science of Fluxions the *Differential Calculus;* but, surely, the differential calculus, if the analogy of language is to be observed, means the calculus of differences.

It was shown (Ch. I. 33.) that the same function admits of successive differentiations; hence a fluxion which is the result of n differentiations may be integrated n times. Symbols for these double, triple fluents are $\int\int$, $\int\int\int$, &c.

Since the two symbols d and $\int$ denote reverse operations, whenever they occur together they destroy each other, thus, $d\int \mathrm{P}dx = \mathrm{P}dx$.

4. *The fluent may sometimes be determined at once from inspection.*

The fluxion of x^3 being $3x^2dx$, the fluent of $3x^2dx$ must be x^3. Also, since $d.x^{m+1} = \overline{m+1}.\,x^m dx$, therefore $\int.\overline{m+1}\;x^m dx = x^{m+1}$. Upon the same principle $\int(ydx + xdy) = xy$, $\int\frac{ydx - xdy}{y^2} = \frac{x}{y}$ and $\int \mathrm{A}a^x dx = a^x$.

It is evident that, in general, if the form of the proposed fluxion is derivable from its fluent by any of the preceding rules, we have only to invert that rule in order to find the fluent. We shall begin with Art. 11. Rule 1.

5. *In Integration we may have to add or to subtract a constant quantity.*

For (Art. 11.) $d(x \pm a) = dx$, therefore the fluent of dx may be either x or $x \pm a$, where a is some constant quantity.

This quantity a, called the *correction*, or the *constant*, is to be determined from the conditions of the question.

6. *If any two corresponding values of the variables be known, we may either eliminate the constant or determine its value.*

Ex. Let $du = \overline{m+1}.\,x^m dx$ $\therefore$ $u = x^{m+1} + \mathrm{C}$ and when $u = r$ let $x = b$ $\therefore$ we have $\quad r = b^{m+1} + \mathrm{C}$

$$\therefore u - r = x^{m+1} - b^{m+1}$$

Also $\mathrm{C} = r - b^{m+1}$.

7. By this correction we can find the value of the fluent included between two known values of its principal variable.

For let it be required to find $\int x^m dx$ between the values of $x = b$ and $x = a$; we have $u = \frac{1}{m+1}.\,x^{m+1} + \mathrm{C}$, and $u = o$ when $x = b$, therefore $o = \frac{1}{m+1}.\,b^{m+1} + \mathrm{C}$,

hence $u = \frac{1}{m+1}.\,(x^{m+1} - b^{m+1})$, and

substituting a for x, we have $(u) = \frac{1}{m+1}.\,(a^{m+1} - b^{m+1})$.

8. A *Definite Integral* is one that has been corrected, and its symbol is $\dot{\int}$. The function may be enclosed in brackets, if the fluent has been corrected, and a particular value assigned to the variable.

9. *The constant multiplier or divisor may be removed from under the symbol* $\int$.

For (Art. 12.) $d.ax = adx$, therefore $\int adx = a\int dx$.

Hence $\int x^m dx = \frac{1}{m+1}.\int \overline{m+1}.\, x^m dx = \frac{x^{m+1}}{m+1} + c.$

Also $\int a^x dx = \frac{1}{\text{A}}.\, a^x$.

We shall, in general, omit the correction; it being understood that it is to be taken into consideration when the form is applied to particular examples.

10. *The fluent of the sum or difference of any number of fluxions is the sum or difference of their fluents.*

For (Art. 13.) $d(u+v-w \pm \&c.) = du+dv-dw \pm \&c.$ therefore $\int(du+dv-dw \pm \&c.) = \int du + \int dv - \int dw \pm \&c. + c$, where c is the result of the corrections of all the fluents.

11. It appears (Art. 14.) that $\int mx^{m-1}dx = x^m$; and from Art. 14. Cor. 3. that $\int n.(a^m+x^m)^{n-1}\, mx^{m-1}dx = (a^m+x^m)^n$: hence, to integrate a compound quantity of the above form, where the part without the vinculum is of the form of the fluxion of the part within, adopt the following:

RULE 1. *Increase the index by unity, divide by the index so increased, and also by the fluxion of the root.*

Examples.

1. $\int (a^m+x^m)^{n-1}x^{m-1}\,dx = \frac{1}{m\,n}\int n.(a^m+x^m)^{n-1}m\,x^{m-1}dx$

$= \frac{1}{m\,n}.\,(a^m+x^m)^n.$

2. $\int(a+x)^3dx = \frac{1}{4}(a+x)^4.$

3. $\int(a^2+x^2)^{\frac{1}{2}}\,xdx = \frac{1}{3}(a^2+x^2)^{\frac{3}{2}}.$

4. $\int(a^2+x^2)^2\,xdx = \frac{1}{6}(a^2+x^2)^3 = u.$

Or thus, $(a^2+x^2)^2xdx = a^4xdx + 2a^2x^3dx + x^5dx$ ∴ (2.10.)

$$u = \frac{a^4x^2}{2} + \frac{a^2x^4}{2} + \frac{x^6}{6} + c.$$

5. $\int(a^4-x^4)^{\frac{5}{3}} \times 3x^3dx = \frac{-9}{32}.(a^4-x^4)^{\frac{8}{3}}.$

6. $\int \frac{x^8 dx}{(a^9 + 6x^9)^{\frac{1}{2}}} = \frac{(a^9 + 6x^9)^{\frac{1}{2}}}{27}$.

7. $\int (a^2x^2 + x^4)^{\frac{1}{2}} (2a^2xdx + 4x^3dx) = \frac{2}{3} \cdot (a^2x^2 + x^4)^{\frac{3}{2}}$.

12. *If the fluxion be a function of two variables, the same rule obtains, provided that the part without the vinculum is of the form of the fluxion of the part within.*

Ex. $\int (xdy + ydx + 2ydy)\,(xy + y^2)^n = \frac{1}{n+1} \cdot (xy + y^2)^{n+1}$.

13. *Fluxions which do not appear under the proper form for integration may sometimes be reduced to it.*

Ex. 1. Let $du = \frac{adx}{x\sqrt{2ax - x^2}}$, required to find u.

$\sqrt{2ax - x^2} = x\sqrt{2ax^{-1} - 1}$ $\therefore$ $du = \frac{adx}{x^2\sqrt{2ax^{-1} - 1}}$

$= ax^{-2}dx \cdot (2ax^{-1} - 1)^{-\frac{1}{2}}$, which is of the proper form,

hence $u = \frac{ax^{-2}dx \cdot (2ax^{-1} - 1)^{\frac{1}{2}}}{\frac{1}{2} \times -2ax^{-2}dx} = -(2ax^{-1} - 1)^{\frac{1}{2}} = -\frac{\sqrt{2ax - x^2}}{x}$.

Ex. 2. $du = \frac{adx}{(a^2 + x^2)^{\frac{3}{2}}} = \frac{adx}{x^3 \cdot (a^2x^{-2} + 1)^{\frac{3}{2}}}$

$= (a^2x^{-2} + 1)^{-\frac{3}{2}} \cdot ax^{-3}dx$ $\therefore$ $u = \frac{(a^2x^{-2} + 1)^{-\frac{1}{2}} \cdot ax^{-3}dx}{-\frac{1}{2} \times -2a^2x^{-3}dx}$

$= \frac{1}{a \cdot (a^2x^{-2} + 1)^{\frac{1}{2}}} = \frac{x}{a\sqrt{a^2 + x^2}}$.

Similarly, $du = \frac{dx\sqrt{a^2 + x^2}}{x^4} = x^{-3}dx\,\sqrt{a^2x^{-2} + 1}$ $\therefore$

$u = -\frac{(a^2 + x^2)^{\frac{3}{2}}}{3a^2x^3}$.

Ex. 3. $du = \frac{dx}{(a^n + x^n)^{\frac{n+1}{n}}} = \frac{dx}{x^{n+1} \cdot (a^n x^{-n} + 1)^{\frac{n+1}{n}}}$

$$=(a^n x^{-n}+1)^{-\frac{n+1}{n}} x^{-n-1}\,dx \therefore u=\frac{(a^n x^{-n}+1)^{-\frac{1}{n}}}{a^n}$$

$$=\frac{x}{a^n.(a^n+x^n)^{\frac{1}{n}}}.$$

Ex. 4. $du=\dfrac{dx}{x^{\frac{n}{2}+1}.\sqrt{a^n+x^n}}=\dfrac{dx}{x^{n+1}\sqrt{a^n x^{-n}+1}}$

$$=(a^n x^{-n}+1)^{-\frac{1}{2}} x^{-n-1}dx, \therefore u=\frac{(a^n x^{-n}+1)^{\frac{1}{2}}}{\frac{1}{2}\times -na^n}$$

$$=\frac{-2\sqrt{a^n+x^n}}{na^n x^{\frac{n}{2}}}.$$

Thus, $\displaystyle\int\frac{adx}{x^2\sqrt{a^2+x^2}}=-\frac{\sqrt{a^2+x^2}}{ax}$,

and $\displaystyle\int\frac{dx}{x^3\sqrt{a^4+x^4}}=-\frac{\sqrt{a^4+x^4}}{a^4x^2}$.

14. *In the application of the preceding Rule, there is one case in which it appears to fail.*

$\displaystyle\int x^m dx=\frac{x^{m+1}}{m+1}+\text{C}$, let $m=-1$, and we have $\displaystyle\int\frac{dx}{x}$ $=\dfrac{x^0}{0}+\text{C}=\infty$; and if, in order to avoid the difficulty, we correct the fluent and suppose $x=b$ when $u=0$, we have

$$u=\frac{x^{m+1}}{m+1}+\text{C}$$

$$\text{and } 0=\frac{b^{m+1}}{m+1}+\text{C},$$

$$\therefore u=\frac{x^{m+1}-b^{m+1}}{m+1};$$

in which expression, if we substitute $m=-1$, the result is $u=\dfrac{0}{0}$. This will be explained Chap. 5.

15. Rule 2. *If the form of the fluxion is such that the numerator is the fluxion of the denominator, its fluent is the hyberbolick logarithm of the denominator.*

Let $\frac{dy}{y}$ represent the form of the fluxion; let $x = ly$, therefore $e^x = y$ and differentiating (1.26.) $e^x dx = dy$, or $dx = \frac{dy}{e^x} = \frac{dy}{y}$; hence $\int \frac{dy}{y} = x = ly$.

If we would express the fluent by a logarithm of any other system than the Naperian, let $x = \text{L}y$, then $a^x = y$, and $\int \frac{dy}{y} = \text{A}x = \frac{\text{L}y}{m}$.

The same conclusion may be deduced from Chap. I. 25; for it is there shown that the logarithms of the same number in different systems are as the moduli of the systems.

In Briggs's system, $\frac{1}{m} = 2.302585093.$ or

$m = .434294481\ldots$ hence $\int \frac{dy}{y} = 2.302585093. \times \log. y.$

Ex. 1. $\int \frac{dx}{x \pm a} = l(x \pm a)$, or $= 2.302585093. \times \log.(x \pm a)$.

Ex. 2. $\int \frac{x^{n-1}\,dx}{a^n + x^n} = \frac{1}{n}.\, l.(a^n + x^n) = l.(a^n + x^n)^{\frac{1}{n}}.$

Ex. 3. $\int \frac{adx}{a + bx} = \frac{a}{b}.\int \frac{dx}{\frac{a}{b} + x} = \frac{a}{b}\, l.\left(\frac{a}{b} + x\right);$

$$\text{or} = \frac{a}{b}\int \frac{bdx}{a + bx} = \frac{a}{b}\, l.(a + bx).$$

Ex. 4. $\int \frac{dx}{1 + x} = l.(1 + x).$

Hence the fluent of $\frac{dx}{1 + x}$, to the value of $x = 1$, is equal to $l2 = 2.302585093. \times \log. 2 = 2.302585093. \times .3010300\ldots. = .69314\ldots..$; to the value of $x = 4$, the fluent $= l5 = 1.6094379\ldots$, and thus, by means of a table of common logarithms, the fluent can be computed for all values of the variable.

There is a very useful table of *hyperbolick* Logarithms at the end of the second volume of T. Simpson's Fluxions. There is also a table given in Hutton's Logarithms, for converting common logarithms into hyperbolick.

16. If in the preceding article we had supposed $x = l.-y$, we should have had, $e^x = -y$ and $dx = \frac{-dy}{-y}$, which, according to the common principles of algebra, $= \frac{dy}{y}$; are we then to infer that $\int \frac{dy}{y} = l. \pm y$?

To explain this it may be observed, that it follows from the nature of logarithms that positive and negative numbers cannot both belong to the same system; thus, if we have the quantities a^4, a^3, a^2, a^1, a^0, a^{-1}, a^{-2}, &c. which decrease in geometrick progression, they can never become negative, though their logarithms may. In the same manner, no term of the negative series, $-a^{-3}$, $-a^{-2}$, $-a^{-1}$, -1, $-a$, $-a^2-$, &c. can ever become positive. It appears then that we cannot change the signs both of the numerator and denominator of the fraction without changing the system.

Thus, $dx = \frac{dy}{y}$ belongs to the positive, and $dx = \frac{-dy}{-y}$ to the system of negative numbers.

As a further illustration take $dx = \frac{dy}{a-y}$, therefore $x = -l(a-y) = l.\frac{1}{a-y}$, which belongs to the positive system, and x is negative because the number is a fraction. But if we change the signs both of the numerator and denominator, we have $dx = \frac{-dy}{y-a}$ and $x = -l(y-a) = l\frac{1}{y-a}$, which is a different number from the former, which shows that if the signs be changed the system is also changed.

17. *Fluxions which do not appear under logarithmick forms may in some cases be made to assume them.*

(1.) $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = l.(x + \sqrt{x^2 \pm a^2})$.

For let $x^2 \pm a^2 = v^2 \therefore xdx = vdv \therefore x : v :: dv : dx$ and $x + v : v :: dx + dv : dx \therefore \frac{dx}{v}$, which $= \frac{dx}{\sqrt{x^2 \pm a^2}}$,

$= \frac{dx+dv}{x+v} \therefore \int \frac{dx}{\sqrt{x^2 \pm a^2}} = l.(x+v) = l.(x+\sqrt{x^2 \pm a^2}).$

(2.) $\int \frac{dx}{\sqrt{x^2 \pm 2ax}} = l.(x \pm a + \sqrt{x^2 \pm 2ax}.)$

Let $x^2 \pm 2ax = v^2 \therefore vdv = (x \pm a)dx \therefore x \pm a : v :: dv : dx$ and $x \pm a + v : v :: dx + dv : dx \therefore \frac{dx}{v}$ or $\frac{dx}{\sqrt{x^2 \pm 2ax}} = \frac{dx+dv}{x \pm a + v}$

$\therefore \int \frac{dx}{\sqrt{x^2 \pm 2ax}} = l.(x \pm a + v) = l.(x \pm a + \sqrt{x^2 \pm 2ax}).$

(3.) $\int \frac{2adx}{a^2 - x^2} = l.\frac{a+x}{a-x}.$

For $\frac{2adx}{a^2-x^2} = \frac{dx}{a+x} - \frac{-dx}{a-x} \therefore \int \frac{2adx}{a^2-x^2} = l(a+x) - l(a-x)$

$= l.\frac{a+x}{a-x}.$

(4.) $\int \frac{2adx}{x^2 - a^2} = l.\frac{x-a}{x+a}.$

For $\frac{2adx}{x^2-a^2} = \frac{dx}{x-a} - \frac{dx}{x+a} \therefore \int \frac{2adx}{x^2-a^2} = l.\frac{x-a}{x+a}.$

(5.) $\int \frac{adx}{x\sqrt{a^2-x^2}} = l.\frac{a-\sqrt{a^2-x^2}}{x}.$

Let $a^2 - x^2 = y^2 \therefore x^2 = a^2 - y^2 \therefore$

$\frac{adx}{x\sqrt{a^2-x^2}} = \frac{axdx}{x^2\sqrt{a^2-x^2}} = a.\frac{-ydy}{(a^2-y^2)y} = -\frac{ady}{a^2-y^2}$, whose fluent by form (3.) is $-\frac{1}{2}l.\frac{a+y}{a-y} = \frac{1}{2}l.\frac{a-y}{a+y} = \frac{1}{2}l.\frac{(a-y)^2}{a^2-y^2}$

$= l.\frac{a-y}{x} = l.\frac{a-\sqrt{a^2-x^2}}{x}.$

(6.) $\int \frac{adx}{x\sqrt{a^2+x^2}} = l.\frac{\sqrt{a^2+x^2}-a}{x}.$

Let $a^2 + x^2 = y^2 \therefore x^2 = y^2 - a^2 \therefore$

$\frac{adx}{x\sqrt{a^2+x^2}} = \frac{axdx}{x^2\sqrt{a^2+x^2}} = \frac{aydy}{(y^2-a^2)y} = \frac{ady}{y^2-a^2} \therefore$ (Ex. 4.),

rad. $= r$, cos. $= \frac{y^2}{r}$; for c $= 0$ if we correct from $y = r$:
and to find the whole value of u between $y = r$ and $y = 0$, we have $u = \frac{r}{4} \times$ quadrant, rad. $= r$.

Ex. 4. To compute the value of $\int \frac{adx}{\sqrt{a^2 - x^2}}$ when $x = \frac{a}{2}$.

The fluent is an arc whose sine is $= \frac{1}{2}$ radius, and is therefore an arc of 30° to rad. $= a$: but an arc of 30° to rad. $1 = .5235 \ldots \therefore (u) = a \times .5235 \ldots$ But the computation is usually carried on by means of logarithms; thus,

Ex. 5. To compute $\int \frac{dx}{1+x^2}$ when $x = 7$.

$u = \tan.^{-1} 7 \therefore$ tab. $\tan. u = 10^{10} \times 7 \therefore \log. \tan. u = 10 + \log. 7 = 10.84509 \ldots \therefore$ from the tables, $u = 81^\circ\, 52'$, or to obtain u in terms of radius 1, we have $360^\circ : 81'\, 52'' :: 2\pi = 6.28..1.42\ldots$ which, therefore, is the required value of $\int \frac{dx}{1+x^2}$.

19. *We may correct a logarithmick fluent by a logarithmick constant;*

$\int \frac{xdx}{a^2+x^2} = \frac{1}{2} l(a^2 + x^2) + \frac{1}{2} l.c^2 = l.c\sqrt{a^2 + x^2}$, in which $\frac{1}{2} l.c^2$ is to be determined from the particular case for which the fluent is to be integrated.

If we suppose the origin of this fluent to be $x = 0$, then c is determined, and we have $\int \frac{xdx}{a^2+x^2} = l.\frac{\sqrt{a^2+x^2}}{a}$.

20. So long as the fluent is *indeterminate*, the correction c is called the *arbitrary constant*, because we may assign to it any value we please: but when we apply the fluent to a particular case, and correct for that case, c ceases to be an *arbitrary* quantity.

Thus $\frac{dx}{a+x} =$, by division, $dx \left\{ \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \text{\&c.} \right\}$,
therefore $\int \frac{dx}{a+x} = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} -$ &c. $+$ c, in which c may be any constant quantity whatever; for if we dif-

ferentiate, the conditions of the equation are satisfied: but if we put $\int \frac{dx}{a+x}$ under the form of $l(a+x)$, we have

$l(a+x) = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} -$ &c. + c, in which c must $= la$, otherwise the equation would not hold good for all the values of x; hence, when $\int \frac{dx}{a+x}$ takes the definite form of $la + \frac{x}{a} - \frac{x^2}{2a^2} +$ &c. the correction can be no longer considered as an arbitrary constant.

Hence also it appears that the series $la + \frac{x}{a} - \frac{x^2}{2a^2} +$ &c. is only one out of the infinite number of values which may be assigned to $\int \frac{dx}{a+x}$.

21. In giving a definite form to the fluent, we are not at liberty to assign any corresponding values we please to the variable and the correction; we are restricted by the conditions of the question, and must select values which are not inconsistent with those conditions; for instance, $\int \frac{dx}{1+x^2}$, which $= \tan.^{-1}x$, =, by division, $-\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} +$ &c. + c, in which if we suppose $x = 0$, we have $0 = \infty +$ c, and, consequently, $\tan.^{-1}x = \infty - \frac{1}{x} + \frac{1}{3x^3} -$&c. whatever be the value of x, which is an absurd result. But if we suppose $x = \infty$, we have $\frac{\pi}{2} =$ c, where π is a quadrantal arc, and, consequently, $\tan.^{-1}x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} +$ &c. which therefore is the required value of $\int \frac{dx}{1+x^2}$.

22. When corresponding values of the fluent and its variable are not given, it is usual to correct the fluent by finding its value as in the preceding article, when $x = 0$, or when $x = \infty$; but there are cases in which neither of these

suppositions will enable us to give a definite form to the fluent. Ex. gr.

$$\frac{dx}{\sqrt{x^2-1}}=\frac{dx}{x}\left(1-\frac{1}{x^2}\right)^{-\frac{1}{2}}$$

$$=\frac{dx}{x}\left\{1+\tfrac{1}{2}\times\frac{1}{x^2}+\tfrac{1}{2}\cdot\tfrac{3}{4}\times\frac{1}{x^4}+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\times\frac{1}{x^6}+\text{\&c.}\right\}$$

$$\therefore\int\frac{dx}{\sqrt{x^2-1}}=lx-\tfrac{1}{2}\times\frac{1}{2x^2}-\tfrac{1}{2}\cdot\tfrac{3}{4}\times\frac{1}{4x^4}-\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\times\frac{1}{6x^6}-\text{\&c.};$$

but $\int\frac{dx}{\sqrt{x^2-1}}$ (17 form (1)), also $=l.(x+\sqrt{x^2-1})$: therefore

$l.(x+\sqrt{x^2-1})=lx-\frac{1}{2}\times\frac{1}{2x^2}-\text{\&c.}+\text{c}$: in which, if we suppose $x=0$ or $x=\infty$, the value of c cannot be obtained; but if we suppose $x=1$, we have

$$\text{c}=\tfrac{1}{2}\cdot\tfrac{1}{2}+\tfrac{1}{2}\cdot\tfrac{3}{4}\times\tfrac{1}{4}+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\times\tfrac{1}{6}+\text{\&c.}$$

23. It may be here observed, in reference to the expansion of $\frac{dx}{1+x^2}$ into a series, that it follows from the nature of symbolical language that the two symbols $\frac{1}{1+x^2}$ and $\frac{1}{x^2+1}$, though numerically equal, are not identical; for they point to different operations. If these operations are performed, there will result two series which are dissimilar in every respect, and which are not even numerically equal, except for certain values of x.

24. Required to expand L$(1+y)$.

It was shown (1.27) that Ly cannot be expanded either in an ascending or a descending series of y; but the same objections are not applicable to the expansion of L$(1+y)$.

Let then $u=\text{L}(1+y)$, therefore $du=\frac{1}{la}\times\frac{dy}{1+y}=$, by division, $\frac{dy}{la}(1-y+y^2-y^3+\text{\&c.})$, therefore, integrating, $u=\text{L}(1+y)=\frac{1}{la}(y-\frac{1}{2}y^2+\frac{1}{3}y^3-\text{\&c.})$, where no correction is necessary.

Cor. 1. $l(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \&c.$

Cor. 2. Hence $ly = (1-y) - \frac{1}{2}(1-y)^2 + \frac{1}{3}(1-y)^3 - \&c.$

24. **The logarithm of a number may be expressed in a series containing positive and negative powers of that number.**

For $y = \dfrac{1+y}{1+\dfrac{1}{y}}$, therefore

$$\text{L}y = \text{L}(1+y) - \text{L}\left(1+\frac{1}{y}\right)$$

$$= \frac{1}{la}\left\{y - y^{-1} - \tfrac{1}{2}(y^2 - y^{-2}) + \tfrac{1}{3}(y^3 - y^{-3}) - \&c.\right\}.$$

This is a more useless series for the purpose of computing logarithms even than the last, as it does not converge in any case.

25. $\text{L}y = \text{M}\left\{\dfrac{y-1}{y} + \tfrac{1}{2}\left(\dfrac{y-1}{y}\right)^2 + \tfrac{1}{3}\left(\dfrac{y-1}{y}\right)^3 + \&c.\right\}.$

For Art. 23, $\text{L}(1+z) = \text{M}\left\{z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \&c.\right\}$

$$\therefore \text{L}(1-z) = -\text{M}\left\{z + \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 + \&c.\right\}$$

Substitute $1 - z = \dfrac{1}{y} \therefore z = \dfrac{y-1}{y}$, and we have

$$\text{L}y = \text{M}\left\{\frac{y-1}{y} + \tfrac{1}{2}\left(\frac{y-1}{y}\right)^2 + \tfrac{1}{3}\left(\frac{y-1}{y}\right)^3 + \&c.\right\}.$$

Since $\text{L}\dfrac{1+z}{1-z} = 2\text{M}\left\{z + \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5 + \&c.\right\}$, if we substitute $\dfrac{1+z}{1-z} = y$ or $z = \dfrac{y-1}{y+1}$, there results

$$\text{L}y = 2\text{M}\left\{\frac{y-1}{y+1} + \tfrac{1}{3}\left(\frac{y-1}{y+1}\right)^3 + \tfrac{1}{5}\left(\frac{y-1}{y+1}\right)^5 + \&c.\right\}.$$

Cor. The sum of either series $1 + \frac{1}{2} + \frac{1}{3} + \&c.$ or $1 + \frac{1}{3} + \frac{1}{5} + \&c.$ *ad infinitum* is infinite.

For if we suppose y to be infinite in the above series, $\text{L}y$ is infinite; and the numerators of the fractions are in the limit equal to the denominators, and, consequently, the sum of the series is in either case infinite.

26. $l(y+n)=ly+2\left\{\frac{n}{2y+n}+\frac{n^3}{3(2y+n)^3}+\frac{n^5}{5(2y+n)^5}+\&c.\right\}$

For it has been shown that $l\frac{1+z}{1-z}=2(z+\frac{1}{3}z^3+\frac{1}{5}z^5+\&c.)$;

substitute $\frac{1+z}{1-z}=1+\frac{n}{y}$, therefore $z=-z+\frac{n.(1-z)}{y}$, or

$2z+\frac{nz}{y}=\frac{n}{y}$ or $z=\frac{n}{2y+n}$; hence, by substitution, $l\left(1+\frac{n}{y}\right)$, which $=l(y+n)-ly$,

$$=2\left\{\frac{n}{2y+n}+\frac{n^3}{3(2y+n)^3}+\frac{n^5}{5(2y+n)^5}+\&c.\right\},$$

and $l(y+n)=ly+2\left\{\frac{n}{2y+n}+\frac{n^3}{3(2y+n)^3}+\frac{n^5}{5(2y+n)^5}+\&c.\right\}$

Cor. 1. $l(y+1)$

$$=ly+2\left\{\frac{1}{2y+1}+\frac{1}{3(2y+1)^3}+\frac{1}{5(2y+1)^5}+\&c.\right\}$$

Cor. 2. $l\left(y+\frac{1}{n}\right)$

$$=ly+2\left\{\frac{1}{2ny+1}+\frac{1}{3(2ny+1)^3}+\frac{1}{5(2ny+1)^5}+\&c.\right\}$$

Cor. 3. If $y+n$ be a large number, such that n being small, $y=a^m$, a convenient series for computing its logarithm is $l(y+n)=mla+2\left\{\frac{n}{a^m+n}+\frac{n^3}{3.(a^m+n)^3}+\&c.\right\}$,

where n may be either positive or negative.

These hyperbolick logarithms may be converted into tabular, by multiplying both sides of the equation by $\frac{1}{l\,10}=.43429448\ldots$ (1. 25.)

The series deduced in this and the preceding article are among the most useful for the computation of logarithms, the latter being required when we would compute the logarithm of the larger number, knowing that of the smaller.

27. The following are examples of the application of these formulæ to all numbers from 2 to 10.

Required $l\,2$.

Here $y=2$ $\therefore \dfrac{y-1}{y+1}=\dfrac{1}{3}$ $\therefore$

$$l\,2=2\left\{\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\&c.\right\}$$

$$\frac{1}{3}=.3333\ 3333.$$

$$\frac{1}{3.3^3}=.0123\ 4567.$$

$$\frac{1}{5.3^5}=.0008\ 2304.$$

$$\frac{1}{7.3^7}=.0000\ 6532.$$

$$\frac{1}{9.3^9}=.0000\ 0564.$$

$$\frac{1}{11.3^{11}}=.0000\ 0051.$$

$$\frac{1}{13.3^{13}}=\frac{.0000\ 0004}{.3465\ 7355}.$$

$\therefore$ $l\,2=.6931\ 4710$, which is accurate to 6 places; and more terms computed give $l\,2=.6931\ 4718\ 05$.

Required $l\,3$.

Here $y=3$ and $\dfrac{y-1}{y+1}=\dfrac{1}{2}$ $\therefore$

$$\frac{1}{2}=.5.$$

$$\frac{1}{3.2^3}=.0416\ 6666\ 66.$$

$$\frac{1}{5.2^5}=.0062\ 5000\ 00.$$

$$\frac{1}{7.2^7}=.0011\ 1607\ 14.$$

$$\frac{1}{9.2^9}=.0002\ 1701\ 38.$$

$$\frac{1}{11.2^{11}}=.0000\ 4438\ 92.$$

$$\frac{1}{13.2^{13}}=.0000\ 0939\ 00.$$

$$\frac{1}{15.2^{15}}=.0000\ 0203\ 45.$$

$$\frac{1}{17.2^{17}}=.0000\ 0044\ 87.$$

$$\frac{1}{19.2^{19}}=.0000\ 0010\ 03.$$

$$\frac{1}{21.2^{21}}=.0000\ 0002\ 27.$$

$$\frac{1}{23.2^{23}}=\frac{.0000\ 0000\ 50}{.5493\ 0614\ 22}\ \therefore$$

$$l\,3=1.0986\ 12\ldots$$

Required $l\,4$.

$$l\,4=2\,.\,l\,2=1.3862\ 94\cdots$$

Required $l\,5$.

$$5=4+1=4\left\{1+\frac{1}{4}\right\}\ \therefore$$

$$l\,5=l\,4+l\left(1+\frac{1}{4}\right)\therefore$$

$$l\,5=l\,4+2\left\{\frac{1}{9}+\frac{1}{3.9^3}+\frac{1}{5.9^5}+\&c.\right\}$$

$$=1.6094\ 37\cdots$$

Required $l\,6$.

$$l\,6=l\,2+l\,3=1.7917\ 59\ldots$$

Required $l\,7$.

$$7 = 8 - 1 = 8\left(1 - \frac{1}{8}\right)$$

$$\therefore l\,7 = 3\,.\,l\,2\,\left\{\frac{1}{8} + \frac{1}{2.8^2} + \frac{1}{3.8^3} + \&c.\right\}$$

$$= 1\cdot9459\ 10\ \cdots$$

Required $l\,8$.

$$l\,8 = 3l\,2 = 2.0794\ 41\ldots$$

Required $l\,9$.

$$l\,9 = 2l\,3 = 2.1972\ 24\cdots$$

Required $l\,10$.

$$l\,10 = l\,5 + l\,2 = \begin{array}{r} 1.6094\ 37 \\ .6931\ 47 \\ \hline 2.3025\ 8\ldots \end{array}$$

28. *Briggs' method of computing logarithms.*

$\text{L}y = \text{M}\{(y - 1) - \frac{1}{2}(y - 1)^2 + \frac{1}{3}(y - 1)^3 - \&c.\}$, and in order to make this series converge, substitute $y^{\frac{1}{n}}$ for y, and we have

$$\text{L}y^{\frac{1}{n}} \text{ or } \frac{1}{n}\text{L}y = \text{M}\left\{(y^{\frac{1}{n}} - 1) - \tfrac{1}{2}(y^{\frac{1}{n}} - 1)^2 + \tfrac{1}{3}(y^{\frac{1}{n}} - 1)^3 - \&c.\right\}$$

$$\text{or } \text{L}y = n\text{M}\left\{(y^{\frac{1}{n}} - 1) - \tfrac{1}{2}(y^{\frac{1}{n}} - 1)^2 + \tfrac{1}{3}(y^{\frac{1}{n}} - 1)^3 - \&c.\right\}.$$

Hence, if the square root or the cube or any other root of y be repeatedly extracted till $y^{\frac{1}{n}} - 1$ becomes a decimal with a great many zeros, the series converges, and we obtain an approximate value of $\text{L}y$.

The process is not so tedious as it may appear to be, from the circumstance that $y^{\frac{1}{n}} - 1$ decreases much faster than n increases.

Since the second term of the series is the square of the first, it is a decimal which contains at least twice as many zeros as the first; hence, if we repeat the operation of extracting the root a sufficient number of times, it is manifest that we may neglect all the terms of the series after the first.

The approximate value thus obtained is obviously greater than the true value.

29. If we substitute $y^{-\frac{1}{n}}$ for y, the series becomes

$$\mathrm{L}y = -n\mathrm{M}\left\{\left(\frac{1}{y^{\frac{1}{n}}}-1\right) - \tfrac{1}{2}\left(\frac{1}{y^{\frac{1}{n}}}-1\right)^2 + \tfrac{1}{3}\left(\frac{1}{y^{\frac{1}{n}}}-1\right)^3 - \&c.\right\}$$

$$= n\mathrm{M}\left\{\left(1-\frac{1}{y^{\frac{1}{n}}}\right) + \tfrac{1}{2}\left(1-\frac{1}{y^{\frac{1}{n}}}\right)^2 + \tfrac{1}{3}\left(1-\frac{1}{y^{\frac{1}{n}}}\right)^3 + \&c.\right\}$$

in which all the terms are positive and converge if y be greater than 1; for then $y^{\frac{1}{n}}$ is greater than 1, and, consequently, 1 is greater than $\frac{1}{y^{\frac{1}{n}}}$.

If the operation of extracting the root be performed as in the preceding article, we shall obtain another approximate value of $\mathrm{L}y$, and this is evidently *less* than the true value.

If then we calculate both values corresponding to the same number n, we shall know to how many places of decimals the result is true.

Cor. The difference between the two approximate values is $n\mathrm{M}\left\{(y^{\frac{1}{n}}-1)-\left(1-\frac{1}{y^{\frac{1}{n}}}\right)\right\} = n\mathrm{M}.\frac{(y^{\frac{1}{n}}-1)^2}{y^{\frac{1}{n}}}$, which therefore diminishes as n increases.

30. *Borda's or Delambre's method.*

It has been shown that $l.\frac{1+z}{1-z} = 2\left\{z+\tfrac{1}{3}z^3+\tfrac{1}{5}z^5+\&c.\right\}$

Assume for $1+z$ and $1-z$ cubicks x^3+qx+r and x^3+qx-r, whose factors are known; let

$$\left.\begin{aligned}1+z&=(x-a)(x-b)(x+a+b)\\1-z&=(x+a)(x+b)(x-a-b)\end{aligned}\right\}\text{theref.}\left\{\begin{aligned}q&=-ab+a^2+b^2\\r&=\ ab.(a+b)\end{aligned}\right\}$$

But $\frac{1+z}{1-z}=\frac{x^3+qx+r}{x^3+qx-r}=\frac{1+\frac{r}{x^3+qx}}{1-\frac{r}{x^3+qx}}$, therefore we have

$$l.\frac{1+z}{1-z} = 2\left\{\frac{r}{x^3+qx}+\tfrac{1}{3}\,\frac{r^3}{(x^3+qx)^3}+\tfrac{1}{5}\,\frac{r^5}{(x^3+qx)^5}+\&c.\right\};$$

or $l(x-a)+l(x-b)+l(x+a+b)-l(x+a)-l(x+b)-$
$l(x-a-b)=2\left\{\frac{r}{x^3+qx}+\frac{1}{3}\frac{r^3}{(x^3+qx)^3}+\frac{1}{5}\frac{r^5}{(x^3+qx)^5}+\text{\&c.}\right\}.$

From which equation $l(x+a+b)$ can be computed, if we know the logarithms of the five preceding numbers, $x-a-b$, $x-a$, $x-b$, $x+b$, $x+a$.

Thus, if we assume

$$1+z=(x-1)(x-1)(x+2)=x^3-3x+2,$$
$$1-z=(x+1)(x+1)(x-2)=x^3-3x-2,$$

we have $l.(x-1)^2+l(x+2)-l.(x+1)^2-l(x-2)$

$$=2\left\{\frac{2}{x^3-3x}+\frac{1}{3}\frac{2^3}{(x^3-3x)^3}+\text{\&c.}\right\}.$$

In which equation if $x=5$, $l\,7$ can be found in terms of $l\,6$, $l\,4$, and $l\,3$; *i. e.* in terms of $l\,2$ and $l\,3$.

Or if the logarithms of 2 and 3 have not been previously computed, substitute for x the numbers 5, 6, 7, 8, successively, and there will result four equations containing the unknown quantities $l\,2$, $l\,3$, $l\,5$, and $l\,7$, which, therefore, may be determined by elimination.

31. *Haros' method.*

This consists in substituting biquadraticks for $1+z$ and $1-z$.

$$\left.\begin{aligned}\text{Assume } 1+z&=(x+3)(x-3)(x+4)(x-4)=x^4-25x^2+144\\ 1-z&=x^2.(x+5)(x-5)=x^4-25x^2\end{aligned}\right\}$$

$$\text{therefore } \frac{1-z}{1+z}=\frac{x^4-qx^2}{x^4-qx^2+r}=\frac{1-\dfrac{\frac{r}{2}}{x^4-qx^2+\frac{r}{2}}}{1+\dfrac{\frac{r}{2}}{x^4-qx^2+\frac{r}{2}}};$$

$$\left.\begin{aligned}\text{therefore}\qquad &2lx+l(x+5)+l(x-5)\\ -l(x+3)-l(x-3)&-l(x+4)-l(x-4)\end{aligned}\right\}$$

$$=2\left\{\frac{72}{x^4-25x^2+72}+\text{\&c.}\right\}.$$

For these and other series which converge even with

greater rapidity, see La Croix's *Introduction Calcul. Diff.* p. 50. The reader may also consult his 3d volume for the method of computing logarithms by interpolation.

32. It may be here observed, that logarithms enable us to generalize the demonstration of the binomial theorem. At least, they enable us to show, that if $(1+x)^n$ is developed in a series of the form $1 + \text{A}x + \text{B}x^2 +$ &c. by any analytical process whatever, A shall $=n$, $\text{B}=n.\frac{n-1}{2}$, &c. = &c. even in the cases where n is either a fraction or an imaginary quantity.

For since $(1+x)^n = 1 + \text{A}x + \text{B}x^2 + \text{C}x^3 +$ &c., therefore $n.l(1+x) = l\{1 + x(\text{A}+\text{B}x+ \&c.)\}$, and by developement,

$$n.\left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.\right\}$$

$$= x(\text{A}+\text{B}x+\&c.) - \frac{x^2.(\text{A}+\text{B}x+\&c.)^2}{2.} + \&c.,$$

and equating like terms (Alg. 346),

$$\text{A} = n,\ \text{B} = n.\frac{n-1}{2},\ \&c. = \&c.$$

When n is a positive integer, the binomial theorem is strictly demonstrable either by the method of induction, or by the method of combinations given in the Algebra (Art. 232), the latter of which is strictly analytical, because it does not assume that the law of the series is known. When n is not a positive integer, the proof given in this article seems quite unexceptionable, since, as La Croix observes, in the developement of $l(1+x)$ we do not assume the expansion of the binomial when n is fractional.

33. *Imaginary functions.*

Imaginary functions are the links which connect circular functions with the exponential and the logarithmick, and as they are of great use in analytical science, we shall consider them in this place.

It appears by induction that every algebraick function of a quantity of the form $a \pm b\sqrt{-1}$ is of the same form. It will be shown that all transcendentals may likewise be expressed in terms of $\sqrt{-1}$, and that they possess the same property as the algebraick functions.

They are not to be regarded as representing the numerical value of quantity, but merely as analytical symbols, which show that by an extension of the common rules of algebra certain quantities may be made to assume certain analytical forms.

If we are sufficiently cautious not to extend these rules beyond what the circumstances of the case will allow, our results will be as accurate and as much to be depended upon as those which are obtained by any other analytical process. We have already seen one instance (Art. 16.) where it was necessary to observe this caution. John Bernoulli contended that because $(-a)^2 = a^2$, therefore $\text{L}.(-a)^2 = \text{L}.(a)^2$ and $2\text{L}(-a) = 2\text{L}(a)$ and $\text{L}(-a) = \text{L}(+a)$; in which he errs at the very first step from not sufficiently attending to the nature of logarithms.

In Art. 35. the developement of $\sin.x$ and $\cos.x$ into series in terms of x will be there assumed; for these series are most conveniently demonstrated by the theorems of the fourth chapter.

In the management of irrational and imaginary functions, it will be generally found convenient to reduce the former to the form of the latter, and we shall then only have to consider the rules for the involution and the evolution of $\sqrt{-1}$.

34. *Bezout's demonstration of the proposition that* $\sqrt{-a} \times \sqrt{-a} = -a$.

$\sqrt{-a} \times \sqrt{-a} = \sqrt{a^2}$, and *generally* $\sqrt{a^2}$ is $\pm a$; but the a^2 in this case has arisen from multiplying $-a$ and not $+a$ into itself, so that $+a$ is excluded by the nature of the case.

Otherwise. $(c-b)a = (c-b)a$, therefore $\sqrt{c-b}\sqrt{a} = \sqrt{(c-b)a}$, which are numerically equal so long as c is greater than b; but suppose $b = c+1$, then we have $\sqrt{-1}\,.\,\sqrt{a} = \sqrt{-a}$, and therefore $\sqrt{-a} \times \sqrt{-a} = (\sqrt{-1}\,.\,\sqrt{a})^2 = -a$. See also Alg. (242.).

Cor. 1. $\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}$.

For $\sqrt{-a} = \sqrt{a}.\sqrt{-1}$, and $\sqrt{-b} = \sqrt{b}\sqrt{-1}$, therefore $\sqrt{-a}.\sqrt{-b} = \sqrt{a}\,.\sqrt{b}\,\sqrt{-1}\,\sqrt{-1} = -\sqrt{ab}$.

Cor. 2. $\sqrt{a} \times \sqrt{-b} = \sqrt{ab}\,.\,\sqrt{-1}$.

Cor. 3. $\dfrac{\sqrt{-a}}{\sqrt{-b}} = \sqrt{\dfrac{a}{b}}$.

Cor. 4. $\dfrac{\sqrt{a}}{\sqrt{-b}} = \dfrac{1}{\sqrt{-1}} \cdot \sqrt{\dfrac{a}{b}}$, or $= -\sqrt{-1}.\sqrt{\dfrac{a}{b}}$.

35. *Involution.*

$$\begin{array}{ll} \sqrt{-1} = +\sqrt{-1} & (\sqrt{-1})^5 = +\sqrt{-1}, \ \&c. = \&c. \\ (\sqrt{-1})^2 = -1 & (\sqrt{-1})^6 = -1 \\ (\sqrt{-1})^3 = -\sqrt{-1} & (\sqrt{-1})^7 = -\sqrt{-1} \\ (\sqrt{-1})^4 = +1 & (\sqrt{-1})^8 = +1. \end{array}$$

But if any integral number be divided by 4, the remainder is either 0, 1, 2, or 3; the four quantities $4n$, $4n+1$, $4n+2$, $4n+3$, by a proper assumption of n may be made to represent the integral numbers.

The following four forms then include all the possible cases:

$$\left\{\begin{array}{ll} (\sqrt{-1})^{4n} = +1) & (\sqrt{-1})^{4n+2} = -1) \\ (\sqrt{-1})^{4n+1} = +\sqrt{-1}) & (\sqrt{-1})^{4n+3} = -\sqrt{-1}) \end{array}\right\}.$$

With respect to the *evolution* of $\sqrt{-1}$; by substituting x for $\sqrt{-1}$ it will appear that its square root $= \pm\sqrt{-1}$; its cube root = the three cube roots of -1, one of which is possible and $= -1$; its biquadratick roots are all impossible, &c. &c.

Ex. 1. $\sqrt[4]{-a}.\sqrt[4]{-b} = \sqrt[4]{ab}\sqrt[4]{-1}\sqrt[4]{-1} = \sqrt[4]{ab}(\sqrt{-1}\sqrt{-1})^{\frac{1}{2}}$
$= \sqrt[4]{ab}\,\sqrt{-1}$.

Ex. 2. $\sqrt[6]{-a}\,\sqrt[6]{-b} = \sqrt[6]{ab}\sqrt[6]{-1}\sqrt[6]{-1} = \sqrt[6]{ab}(\sqrt{-1}\sqrt{-1})^{\frac{1}{3}}$
$= \sqrt[6]{ab} \times (-1)^{\frac{1}{3}}$, which has three values, one of which is possible and $= -\sqrt[6]{ab}$.

36. *The trigonometrical lines may all be expressed in terms of* $\sqrt{-1}$.

For, since $e^x = 1 + \dfrac{x}{1} + \dfrac{x^2}{1.2} + \dfrac{x^3}{1.2.3} + \dfrac{x^4}{1.2.3.4}$
$+ \dfrac{x^5}{1.2.3.4.5} + \&c.$, for x substitute $x\sqrt{-1}$, therefore we have

$$e^{x\sqrt{-1}}$$

$$=1+\frac{x\sqrt{-1}}{1}-\frac{x^2}{1.2}-\frac{x^3\sqrt{-1}}{1.2.3}+\frac{x^4}{1.2.3.4}+\frac{x^5\sqrt{-1}}{1.2.3.4.5}-\&c.$$

$$=1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\&c.$$

$$+\sqrt{-1}\left\{\frac{x}{1}-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\&c.\right\}$$

$=\cos.x+\sqrt{-1}\sin.x.$ Vid. ch. 4, 3. Ex. 4.

Similarly $e^{-x\sqrt{-1}}=\cos.x-\sqrt{-1}\sin.x$; therefore, adding and subtracting,

$$\cos.x=\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2}, \text{ and } \sin.x=\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}}.$$

Otherwise.

Let $y=\cos.x$, therefore (1. 32.) $dx=\frac{-dy}{\sqrt{1-y^2}}$, and $\frac{-dx}{\sqrt{-1}}$, which $=\sqrt{-1}.dx$, $=\frac{dy}{\sqrt{y^2-1}}$, therefore $x\sqrt{-1}$ $=l(y+\sqrt{y^2-1})$, which requires no correction; hence $e^{x\sqrt{-1}}=y+\sqrt{y^2-1}$, and $e^{2x\sqrt{-1}}-2e^{x\sqrt{-1}}y=-1$, and

$$y=\frac{e^{2x\sqrt{-1}}+1}{2e^{x\sqrt{-1}}}=\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2}.$$

Also, $\sin.x=\sqrt{1-\cos.^2x}=\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}}$.

Since all the trigonometrical lines may be expressed in terms of sine and cosine, they may be expressed in terms of $\sqrt{-1}$.

Thus $\tan.x=\frac{\sin.x}{\cos.x}=\frac{1}{\sqrt{-1}}\cdot\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}$

$$=\frac{1}{\sqrt{-1}}\cdot\frac{e^{2x\sqrt{-1}}-1}{e^{2x\sqrt{-1}}+1}.$$

Ex. 1. To show that $2\sin.x\cos.x = \sin.2x$.

For multiplying the symbols into each other, we have

$$\sin.x\cos.x = \frac{e^{2x\sqrt{-1}} - e^{-2x\sqrt{-1}}}{4\sqrt{-1}}$$

$$\therefore 2\sin.x\cos.x = \frac{e^{2x\sqrt{-1}} - e^{-2x\sqrt{-1}}}{2\sqrt{-1}} = \sin.2x.$$

Ex. 2. To show that $x = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \&c.$ where $t = \tan.x$.

For since $e^{x\sqrt{-1}} = \cos.x + \sqrt{-1}\sin.x$, therefore

$x\sqrt{-1} = l(\cos.x + \sqrt{-1}\sin.x)$; and similarly

$-x\sqrt{-1} = l(\cos.x - \sqrt{-1}\sin.x)$,

$$\text{therefore } 2x\sqrt{-1} = l\frac{\cos.x + \sqrt{-1}\sin.x}{\cos.x - \sqrt{-1}\sin.x}$$

$$= l.\frac{1 + \sqrt{-1}\tan.x}{1 - \sqrt{-1}\tan.x} =$$

$$2\{\sqrt{-1}\tan.x + \tfrac{1}{3}(\sqrt{-1}\tan.x)^3 + \tfrac{1}{5}(\sqrt{-1}\tan.x)^5 + \&c.\}$$

therefore $x = \tan.x - \frac{1}{3}\tan.^3x + \frac{1}{5}\tan.^5x - \&c.$

Cor. Hence $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \&c.$

Ex. 3. To show that $\frac{x}{2} = \frac{\sin.x}{1} - \frac{\sin.2x}{2} + \frac{\sin.3x}{3} - \&c.$

For (24.) $ly = \{(y - y^{-1}) - \frac{1}{2}(y^2 - y^{-2}) + \frac{1}{3}(y^3 - y^{-3}) - \&c.\}$ in which substitute $e^{x\sqrt{-1}}$ for y, and there results

$$x\sqrt{-1} = \left\{(e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}) - \tfrac{1}{2}(e^{2x\sqrt{-1}} - e^{-2x\sqrt{-1}}) + \tfrac{1}{3}(e^{3x\sqrt{-1}} - e^{-3x\sqrt{-1}}) - \&c.\right\},$$

therefore $\frac{x}{2} = \sin.x - \frac{1}{2}\sin.2x + \frac{1}{3}\sin.3x - \&c.$

Ex. 4. The series of art. 24. also leads to a remarkable analytical expression for the quadrant of a circle, which was first given by John Bernoulli; $(\sqrt{-1})^{\sqrt{-1}} = e^{-\frac{\pi}{2}}$).

For substitute $y = \sqrt{-1}$, and there results

$l.\sqrt{-1}=$

$$\left\{\left(\sqrt{-1}-\frac{1}{\sqrt{-1}}\right)-\tfrac{1}{2}(-1+1)+\tfrac{1}{3}\left(-\sqrt{-1}+\frac{1}{\sqrt{-1}}\right)-\&c.\right\}$$

$$=\frac{-2}{\sqrt{-1}}\left\{1-\tfrac{1}{3}+\tfrac{1}{5}-\&c.\right\}=\frac{-1}{\sqrt{-1}}\cdot\frac{\pi}{2},$$

and $(\sqrt{-1})^{\sqrt{-1}}=e^{-\frac{\pi}{2}}$.

Otherwise.

Since $x\sqrt{-1}=l(\cos.x+\sqrt{-1}\sin.x)$, suppose $x=\frac{\pi}{2}$ or $\cos.x=0$, and there results $\frac{\pi}{2}\sqrt{-1}=l.\sqrt{-1}\ \therefore\ \sqrt{-1}$ $=e^{\frac{\pi}{2}\sqrt{-1}}\ \therefore (\sqrt{-1})^{\sqrt{-1}}=e^{-\frac{\pi}{2}}=.20789\ldots$

37. *De Moivre's Formula.*

$$\left\{\begin{array}{l}(\cos.x+\sqrt{-1}\sin.x)^n=\cos.nx+\sqrt{-1}\sin.nx\\(\cos.x-\sqrt{-1}\sin.x)^n=\cos.nx-\sqrt{-1}\sin.nx\end{array}\right\}$$

For $\cos.x+\sqrt{-1}\sin.x=e^{x\sqrt{-1}}$ (art. 36.), therefore $\cos.nx+\sqrt{-1}\sin.nx=e^{nx\sqrt{-1}}=(\cos.x+\sqrt{-1}\sin.x)^n$.

And since the possible parts of this equation are equal, and also the impossible parts, we have $\cos.nx-\sqrt{-1}\sin.nx$ $=(\cos.x-\sqrt{-1}\sin.x)^n$. (Alg. 256 and 257.)

Cor. Substitute $\alpha=e^{x\sqrt{-1}}$, therefore $\frac{1}{\alpha}=e^{-x\sqrt{-1}}$, hence

$$\left.\begin{array}{l}\text{since } \cos.x+\sqrt{-1}\sin.x=\alpha\\ \text{and } \cos.x-\sqrt{-1}\sin.x=\frac{1}{\alpha}\end{array}\right\},\ \text{therefore } 2\cos.x=\alpha+\frac{1}{\alpha},$$

therefore $2\cos.nx$, which $=e^{nx\sqrt{-1}}+e^{-nx\sqrt{-1}},=\alpha^n+\frac{1}{\alpha^n}$.

38. *Required to develope the sine and cosine of the multiple arc in terms of the powers of the sine and cosine of the simple arc.*

(De Moivre's formula).

$\cos.nx + \sqrt{-1}\sin.nx = (\cos.x + \sqrt{-1}\sin.x)^n =$ by developement,

$$\cos.^n x + n\sqrt{-1}\cos.^{n-1}x\sin.x - n.\frac{n-1}{2}\cos.^{n-2}x\sin.^2x - \&c.$$

therefore (Alg. 253.)

$$(\alpha)\ \cos.nx = \cos.^n x - n.\frac{n-1}{2}\cos.^{n-2}x\sin.^2x$$

$$+n.\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cos.^{n-4}x\sin.^4x - \&c.$$

and dividing by $\sqrt{-1}$,

$$(\beta)\ \sin nx = n\cos.^{n-1}x\sin.x - n.\frac{n-1}{2}\cdot\frac{n-2}{3}\cos.^{n-3}x\sin.^3x - \&c.$$

This series terminates whenever n is a positive integer.

Cor. Hence $\tan.nx$ may be developed.

For $\tan.nx = \dfrac{\sin.nx}{\cos.nx}$

$$= \frac{n\cos.^{n-1}x\sin.x - n.\frac{n-1}{2}\cdot\frac{n-2}{3}\cos.^{n-3}x\sin.^3x + \&c.}{\cos.^n x - n.\frac{n-1}{2}\cos.^{n-2}x\sin.^2x + \&c.},$$

$$= \frac{n\tan.x - n.\frac{n-1}{2}\cdot\frac{n-2}{3}\tan.^3x + \&c.}{1 - n.\frac{n-1}{2}\tan.^2x + \&c.}.$$

39. The series for $\sin.nx$ and $\cos.nx$ deduced in the preceding article would be more commodious if they did not contain combinations of $\sin.x$ and $\cos.x$.

Now $\sin.x$ may always be eliminated from the series (α) by substituting $1 - \cos.^2x$ for $\sin.^2x$; and since $\sin.x$ is in each term raised to an even power, the expression for $\cos.nx$ is in all cases rational. But if we eliminate $\cos.x$ from (β) by substituting $\cos.x = \sqrt{1-\sin.^2x}$, it is rational only when n is an *odd* number.

Euler has given the following series for the sine and cosine of a multiple arc.

Let $p = 2\cos.x$, then

(γ) $2\cos.nx$

$$=p^n-\frac{n}{1}p^{n-2}+\frac{n.(n-3)}{1.2}p^{n-4}-\frac{n.(n-4)(n-5)}{1.2.3}p^{n-6}$$

$$+\frac{n.(n-5)(n-6)(n-7)}{1.2.3.4}p^{n-8}-\&c.$$

(δ) $\sin.nx$

$$=\sin.x\Big\{p^{n-1}-\frac{n-2}{1}p^{n-3}+\frac{(n-3)(n-4)}{1.2}p^{n-5}$$

$$-\frac{(n-4)(n-5)(n-6)}{1.2.3}p^{n-7}+\frac{(n-5)(n-6)(n-7)(n-8)}{1.2.3.4}p^{n-9}$$

$$-\&c.\Big\};$$

of which the first shall be demonstrated in the fourth chapter, and the second may be deduced by differentiating the first and dividing by $2ndx$. For other series, see the Trigonometry, App. pp. 241.

40. *Given the cosine of an arc to a certain radius, required the cosine of an arc of the* same magnitude *to any other radius.*

Let A^0 and x^0 be the two angles which an arc of the same magnitude subtends, having the radii R and r respectively; then, since $\angle \propto \frac{\text{arc.}}{\text{rad.}} \propto$, in this case, $\frac{1}{\text{rad.}}$, therefore $x=\frac{A.R}{r}$, or x is a multiple of A, and, consequently, its trigonometrical cosine may be found in terms of cos.A by means of De Moivre's formula, and we have $(\cos.A+\sqrt{-1}\sin.A)^{\frac{R}{r}}=\cos.x+\sqrt{-1}\sin.x$, or $\cos.x+\sqrt{\cos.^2x-1}$

$$=(\cos.A+\sqrt{\cos.^2A-1})^{\frac{R}{r}}=\frac{1}{R^{\frac{R}{r}}}.(R.\cos.A+\sqrt{R^2\cos.^2A-R^2})^{\frac{R}{r}},$$

from which equation cos.x may be had in terms of R cos.A; let $\cos.x=q\,.\,R\cos.A$, then we have $r\cos.x=rq\,.\,R\cos.A$, which is the required cosine in terms of the given cosine.

41. *The logarithm of an imaginary quantity may have an infinite number of analytical values.*

For let $a\pm b\sqrt{-1}$ be the quantity; take $a=r\cos.x$, $b=r\sin.x$, and, consequently, $a^2+b^2=r^2$; then

$$\text{L}(a \pm b\sqrt{-1}) = \text{L}r + \text{L}(\cos.x \pm \sqrt{-1}\sin.x),$$
$$= \text{L}r \pm x\sqrt{-1}, \text{ (art. 36.)}$$

In this equation we may suppose x to become either $2\pi + x$, $4\pi + x \cdots$ or $2i\pi + x$, where i is any integer, which will not alter the values of a, b, and r, and, consequently, we may have

$$\begin{aligned}\text{L}(a \pm b\sqrt{-1}) &= \text{L}r \pm x\sqrt{-1},\\ \text{or } &= \text{L}r \pm (2\pi + x)\sqrt{-1},\\ \text{or } &= \text{L}r \pm (4\pi + x)\sqrt{-1},\\ \text{or } &= \&\text{c.}\\ &= \text{L}r \pm (2i\pi + x)\sqrt{-1}.\end{aligned}$$

42. *The logarithm of a quantity which is not imaginary, possesses an infinite number of analytical values, only one of which is real.*

For in the preceding formula, suppose $x = 0$, then $b = 0$ and $a = r\cos.x = r$, and the general form becomes $\text{L}a = \text{L}a \pm 2i\pi\sqrt{-1}$ where i may be any integer whatever, which shows that $\text{L}a$ may be put under an infinite number of forms, all of which are imaginary, except the one that arises from supposing $i = 0$.

For the sake of convenience $\sqrt[n]{a^n}$ is frequently considered to be the same as a multiplied into the n roots of unity; upon the same principle of convenience, if we suppose $\text{L}a = \text{L}(a + 1) = \text{L}a + \text{L}1$ we may consider $0, 2\pi\sqrt{-1}$, $4\pi\sqrt{-1}$, &c. as the logarithms of unity, or $\text{L}.1 = 2i\pi\sqrt{-1}$.

And in order to distinguish the analytical value of $\text{L}a$ from its real or numerical value, place a dash over the latter, and we shall have $\text{L}a = \text{L}'a + \text{L}1$.

43. *The sum of any two of the analytical values of the logarithms of two quantities is equal to an analytical value of the logarithm of their product.*

$$\begin{aligned}\text{For let } \text{L}a &= \text{L}'a + 2i\pi\sqrt{-1},\\ \text{L}b &= \text{L}'b + 2i'\pi\sqrt{-1},\end{aligned}$$

$$\text{therefore } \text{L}a + \text{L}b = \text{L}'a + \text{L}'b + 2(i + i')\sqrt{-1}.$$
$$\begin{aligned}\text{Also } \text{L}(ab) &= \text{L}'(ab) + 2i''\sqrt{-1}\\ &= \text{L}'a + \text{L}'b + 2i''\sqrt{-1}.\end{aligned}$$

Hence, whichever of the values of $\text{L}a$ and of $\text{L}b$ be taken, even if they are not corresponding values, their sum is equal

to one of the values of $\text{L}(ab)$, which value may be found by taking $i'' = i + i'$.

In the same manner it may be shown that the difference of any two values of $\text{L}a$ and of $\text{L}b$ is equal to one of the values of $\text{L}\,.\,\dfrac{a}{b}$.

44. *All the values of the logarithm of a negative number in a positive system are imaginary.*

For since (41.) $\text{L}(a \pm b\sqrt{-1}) = \text{L}r \pm (2i\pi + x)\sqrt{-1}$; substitute $x = 180^\circ = \pi$, and, consequently, $b = 0$ and $a = -1$ and $r = \sqrt{a^2 + b^2} = (34.) - a = -1$, hence we have $\text{L}(-1) = \text{L}'(-1) \pm (2i\pi + 1)\sqrt{-1}$, which must be imaginary, whatever be the value of i.

Negative numbers then can find no place in a system of positive numbers, and *vice versâ.*

45. *It was seen (art. 36.) that the sines, cosines, &c. of real arcs may be put under imaginary forms. It is true also of the inverse functions, that the arcs of real sines and cosines may be made to take imaginary forms.*

For let the proposed function be $\sin.(a \pm b\sqrt{-1})$.

Then $\sin.(a \pm b\sqrt{-1}) = \sin.a\cos.b\sqrt{-1} \pm \cos.a\sin.b\sqrt{-1}$;

$$\text{but } \cos.x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}, \text{ and}$$

$$\sin.x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}},$$

in which, if we substitute $x = b\sqrt{-1}$ or $x\sqrt{-1} = -b$, we have $\cos.b\sqrt{-1} = \dfrac{e^{-b} + e^{b}}{2}$ and

$$\sin.b\sqrt{-1} = \frac{e^{-b} - e^{b}}{2\sqrt{-1}} = \sqrt{-1}\,.\,\frac{e^{b} - e^{-b}}{2},$$

and, consequently, $\sin.(a \pm b\sqrt{-1})$

$$= \frac{e^{b} + e^{-b}}{2}\sin.a \pm \sqrt{-1}\,.\,\frac{e^{b} - e^{-b}}{2}\,.\cos.a.$$

Similarly $\cos.(a \pm b\sqrt{-1})$

$$= \frac{e^{b} + e^{-b}}{2}\,.\cos.a \mp \sqrt{-1}\,\frac{e^{b} - e^{-b}}{2}\sin.a.$$

In the same manner analytical expressions may be deduced for all the other circular functions of $a \pm b\sqrt{-1}$.

Now in these expressions suppose $\cos. a = 0$, or $a = \frac{\pi}{2}$ or $= \frac{3\pi}{2}$ or $=$ &c. $= (2i+1)\frac{\pi}{2}$; then $\sin. a = \pm 1$, and we have

$$\sin.\left((2i+1)\frac{\pi}{2} \pm b\sqrt{-1}\right) = \pm \frac{e^b + e^{-b}}{2},$$

$$\text{or } \sin.^{-1}.\left(\pm \frac{e^b + e^{-b}}{2}\right) = (2i+1)\frac{\pi}{2} \pm b\sqrt{-1}.$$

In the same manner an arc whose cosine or tangent does not contain $\sqrt{-1}$ may in some cases be expressed by a formula which necessarily includes $\sqrt{-1}$.

Since $\frac{e^b + e^{-b}}{2} = \frac{e^{2b} + 1}{2e^b}$ is always greater than 1 so long as b is finite, we see the reason why $\sin.^{-1} \frac{e^b + e^{-b}}{2}$ and $\cos.^{-1} \frac{e^b + e^{-b}}{2}$ prove to be impossible for all values of i.

For further details on this curious and interesting subject, La Croix refers to Euler's Memoir published in the *Novi Commentarii Academiæ Petrop.* which I have not had an opportunity of consulting.

45. *Fluents may sometimes be integrated by Division.*

Ex. 1. $du = \frac{x^3 dx}{a - x}$.

$$du = \frac{x^3 dx}{-x+a} =, \text{ by division, } -x^2dx - axdx - a^2dx + \frac{a^3dx}{a-x}$$

$$\therefore u = -\frac{x^3}{3} - \frac{ax^2}{2} - a^2x - a^3 l(a - x).$$

Generally, if $du = \frac{x^p dx}{x \pm a}$ where p is a positive integer, the series will always terminate, and the fluent may be integrated; and when the denominator is $x + a$, the signs of the terms will be alternately $+$ and $-$.

Ex. 2. $du = \frac{x^p dx}{a + bx^n}$.

By division,

$$du = \frac{1}{b}\left\{ x^{p-n}dx - \frac{a}{b}x^{p-2n}dx + \frac{a^2}{b^2}x^{p-3n}dx - \text{\&c.} \right\}$$

$$\therefore u = \frac{1}{b}\left\{\frac{x^{p-n+1}}{p-n+1} - \frac{a}{b}\cdot\frac{x^{p-2n+1}}{p-2n+1} + \frac{a^2}{b^2}\cdot\frac{x^{p-3n+1}}{p-3n+1} - \&c.\right\}.$$

This series will not terminate unless $p + 1$ is a positive multiple of n, in which case du may be always reduced to the form of $\int\frac{x^{n-1}dx}{a+bx^n}$.

Ex. 3. $du = \dfrac{dx}{x+2ax^2+x^3}$.

$$du = \frac{x^{-1}dx}{1+2ax+x^2} =\text{, by division, } x^{-1}dx - \frac{2adx+xdx}{1+2ax+x^2}$$

$$\therefore u = l\frac{x}{(1+2ax+x^2)^{\frac{1}{2}}} - \int\frac{adx}{1+2ax+x^2}.$$

PRAXIS.

1. $\int\frac{x^2dx}{a+x} = \frac{x^2}{2} - ax + a^2l(a+x)$.

2. $\int\frac{x^2dx}{a-x} = -\frac{x^2}{2} - ax - a^2l(a-x)$.

3. $\int\frac{x^2dx}{x-a} = \frac{x^2}{2} + ax + a^2l(x-a)$.

4. $\int\frac{x^{3n-1}dx}{a^n-x^n} = -\frac{1}{2n}\left\{x^{2n} + 2a^nx^n + 2a^{2n}l(a^n-x^n)\right\}$.

5. $\int\frac{a-by^2}{a+cy^2}.ydy = -\frac{by^2}{2c} + \frac{ac+ab}{2c^2}\,l\left(y^2+\frac{a}{c}\right)$.

46. *If in a binomial fluent the index of the variable without the vinculum increased by unity is a positive multiple of the index of the variable under the vinculum, the integration can be effected.*

For let $(a + bx^n)^m x^{rn-1}dx$ be the proposed form; first suppose m to be a positive integer, then the binomial may be expanded in finite terms, each of which can be integrated, whatever be the value of r; but if m is not a positive integer, substitute $y = a + bx^n \therefore x^n = \frac{1}{b}(y-a) \therefore$

$$x^{rn} = \frac{1}{b^r}(y-a)^r \therefore x^{rn-1}dx = \frac{1}{nb^r}(y-a)^{r-1}dy \therefore$$

$$du = \frac{1}{nb^r}(y - a)^{r-1}y^m dy$$

$$= \frac{1}{nb^r}\left\{y^{m+r-1}dy - (r-1)ay^{m+r-2}dy + \frac{(r-1)(r-2)}{1.2}a^2y^{m+r-3}dy - \&c.\right\}$$

$$\therefore u = \frac{1}{nb^r}\left\{\frac{y^{m+r}}{m+r} - \frac{(r-1)ay^{m+r-1}dy}{m+r-1} + \frac{(r-1)(r-2)a^2y^{m+r-2}}{1.2(m+r-2)} - \&c.\right\}.$$

Since r is a positive integer, the series arising from the expansion of $(y - a)^{r-1}$ will terminate, and the fluent therefore can be integrated whatever be the values of m and n.

In the case in which m is a negative integer equal to or less than r, one of the terms of the series will be a logarithm, and the fluent cannot be obtained in *algebraick* terms.

Cor. 1. du may be put under the form . . . . $(ax^{-n}+b)^m x^{(m+r)n-1}dx$, and if this be transformed as in the article, it will appear that du may likewise be integrated when the sum of m and r make up a *negative* integer.

Cor. 2. If r is a negative integer, or if $m+r$ is a positive integer, in either case the fluent is integrable by a method which will be given in art. 55.

Cor. 3. If all the preceding conditions of integrability fail, the binomial may be expanded in an infinite series; and if it converges, an approximation may be made to the value of the fluent.

Ex. 1. $du = \dfrac{xdx}{(a+x)^{\frac{1}{2}}}$.

Here $r = 2$ and $u = \frac{1}{3}(a + x)^{\frac{1}{2}}(2x - 4a)$.

Ex. 2. $du = \dfrac{x^{3n-1}dx}{\sqrt{a+bx^n}}$

$$\therefore u = \frac{(a + bx^n)^{\frac{1}{2}}}{15nb^3} \times (6b^2x^{2n} - 8abx^n + 16a^2).$$

Ex. 3. $du = \dfrac{x^3dx}{(x-a)^3}$.

Here $r = 4$ and $m = -3$, and $u = x - a + 3al(x-a)$
$-\frac{3a^2}{x-a}-\frac{a^3}{2.(x-a)^2}$.

Ex. 4. $du = \frac{\sqrt{y^2+b^2}.dy}{y^6}$.

Here $m + r_1 = -2$, and $u = \frac{(b^2+y^2)^{\frac{3}{2}}(2y^2-3b^2)}{15b^4y^5}$.

Ex. 5. $du = (a - bx^n)^{\frac{1}{2}} \cdot x^{-\frac{7n}{2}-1}dx$

$\therefore u = -\frac{(a-bx^n)^{\frac{3}{2}}}{105na^3.x^{\frac{7n}{2}}}.(30a^2 + 24abx^n + 16b^2x^{2n})$.

PRAXIS.

1. $\int \frac{dx}{x^2\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x}$.

2. $\int \frac{a^5dx}{x^4\sqrt{ax+x^2}} = \int \frac{a^5dx}{x^{\frac{9}{2}}\sqrt{a+x}}$, where $m + r = -4$.

3. $\int \frac{dx\sqrt{ax+x^2}}{x^5} = \frac{-30a^2+24ax-16x^2}{105a^3x^{\frac{7}{2}}}.(a+x)^{\frac{3}{2}}$.

4. $\int x^2dx\sqrt{a+x}$
$= 2(a+x)^{\frac{3}{2}}\left\{\frac{(a+x)^2}{7}-\frac{2a}{5}(a+x)+\frac{a^2}{3}\right\}$.

5. $\int \frac{x^{\frac{1}{3}}dx}{(a-bx^{\frac{2}{3}})^{\frac{3}{5}}} = \frac{-15}{28b^2}(a-bx^{\frac{2}{3}})^{\frac{2}{5}}(5a+2bx^{\frac{2}{3}})$.

47. *There are other substitutions which frequently facilitate the integration.*

Ex. 1. $du = \frac{x^{\frac{1}{2}n-1}dx}{a^n+x^n}$.

Substitute $a^n = b^2$ and $x^n = y^2$ $\therefore x^{\frac{1}{2}n-1}dx = \frac{2}{n}dy$ $\therefore$

$du = \frac{2}{n}\cdot\frac{dy}{b^2+y^2}$ $\therefore u = \frac{2}{nb}\tan.^{-1}\frac{y}{b}$.

$$=\frac{2}{na^{\frac{n}{2}}}\tan.^{-1}.\left(\frac{x}{a}\right)^{\frac{n}{2}} \text{ (18 form (3)).}$$

Ex. 2. $du = \dfrac{x^{\frac{1}{2}n-1}dx}{a^n - x^n}$.

By the same substitutions, $du = \dfrac{2}{n}\cdot\dfrac{dy}{b^2-y^2}$ $\therefore$

(17 form (3)) $u = \dfrac{1}{nb}l.\dfrac{b+y}{b-y} = \dfrac{1}{na^{\frac{n}{2}}}l.\dfrac{a^{\frac{n}{2}}+x^{\frac{n}{2}}}{a^{\frac{n}{2}}-x^{\frac{n}{2}}}$.

Ex. 3. $du = \dfrac{x^{\frac{1}{2}n-1}dx}{\sqrt{a^n+x}}$.

Here $du = \dfrac{2}{n}\cdot\dfrac{dy}{\sqrt{b^2+y^2}}$ $\therefore$ (17 form (1))

$u = \dfrac{2}{n}.l(y+\sqrt{b^2+y^2}) = \&c.$

Ex. 4. $du = \dfrac{x^{\frac{1}{2}n-1}dx}{\sqrt{a^n-x^n}}$.

Here $du = \dfrac{2}{n}\cdot\dfrac{dy}{\sqrt{b^2-y^2}}$

$\therefore u = \dfrac{2}{n}\sin.^{-1}.\dfrac{y}{b}. = \&c.$ (18 form (1)).

Ex. 5. $du = \dfrac{dx}{ax^2 \pm bx + c}$.

$du = \dfrac{1}{a}\times\dfrac{dx}{x^2\pm\dfrac{b}{a}x+\dfrac{c}{a}}$; substitute $y = x \pm \dfrac{b}{2a}$

$\therefore x^2 \pm \dfrac{b}{a}x + \dfrac{c}{a} = y^2 + \dfrac{c}{a} - \dfrac{b^2}{4a^2} = y^2 + \dfrac{4ac-b^2}{4a^2} =$, by substitution, $y^2 \pm k^2$ according as the roots of the equation $ax^2 \pm bx + c = 0$ are impossible or possible: whence

$du = \dfrac{1}{a}\cdot\dfrac{dy}{y^2\pm k^2}$, which is either a tangential or a logarithmick form.

Ex. 6. $du = \dfrac{x^{n-1}dx}{ax^{2n} \pm bx^n + c}$.

Substitute $y = x^n \therefore du = \dfrac{\frac{1}{n} \cdot dy}{ay^2 \pm by + c}$, which is integrated in the preceding example.

Ex. 7. $du = \dfrac{dx}{\sqrt{ax^2 \pm bx + c}}$.

Substitute $y = x \pm \dfrac{b}{2a} \therefore du = \dfrac{1}{\sqrt{a}} \times \dfrac{dy}{\sqrt{y^2 \pm k^2}}$, where $k^2 = \dfrac{4ac - b^2}{4a^2}$, which are elementary forms.

Ex. 8. $du = \dfrac{x^{n-1}dx}{\sqrt{ax^{2n} \pm bx^n + c}}$.

Substitute $y = x^n \therefore du = \dfrac{\frac{1}{n} \cdot dy}{\sqrt{ay^2 + by + c}}$, which has been integrated.

Ex. 9. $du = \dfrac{x^p dx}{ax^2 \pm bx + c}$.

Substitute $y = x \pm \dfrac{b}{2a} \therefore du = \dfrac{\left(y \mp \frac{b}{2a}\right)^p dy}{y^2 \pm k^2}$, and if p is a positive integer, the expansion of $\left(y \mp \dfrac{b}{2a}\right)^p$ will terminate, and each term may be integrated by division.

Ex. 10. $du = \dfrac{x^p dx}{ax^{2n} + bx^n + c}$.

Substitute $y = x^n \therefore x^{p+1} = y^{\frac{p+1}{n}} \therefore x^p dx = \dfrac{1}{n} y^{\frac{p+1}{n} - 1} dy$,

$\therefore du = \dfrac{\frac{1}{n} y^{\frac{p+1}{n} - 1} dy}{ay^2 + by + c}$, which can be integrated when $\dfrac{p+1}{n}$ is a positive integer.

Ex. 11. $du = \dfrac{\alpha x^{n-1}dx + \beta x^{2n-1}dx + \gamma x^{3n-1}dx + \delta x^{4n-1}dx}{(a + bx^n + cx^{2n})^p}$

may be integrated by the same method as the preceding example; the trinomial being reduced to a binomial by taking away its second term (Alg. 284).

Ex. 12. $du = \dfrac{axdx + bdx}{x^2 - px + q}$.

Substitute $y = x - \dfrac{p}{2}$ $\therefore x = y + \dfrac{p}{2}$ and $dx = dy$ $\therefore$

$du = \dfrac{aydy + rdy}{y^2 \pm k^2}$, where $r = \dfrac{ap}{2} + b$ and $k^2 = q - \dfrac{p^2}{4}$,

$\therefore u = \dfrac{a}{2}\, l(y^2 \pm k^2) + r\int \dfrac{dy}{y^2 \pm k^2}$, a tangential or a logarithmick form according as the roots of the equation $x^2 - px + q = o$ are impossible or possible.

Ex. 13. $du = \dfrac{dx}{\sqrt{a + x - \sqrt{a^2 + x^2}}}$.

Substitute $y = \sqrt{a + x - \sqrt{a^2 + x^2}}$ $\therefore a^2 + x^2 = (a + x)^2 - 2y^2.(a + x) + y^4$ $\therefore 0 = 2ax - 2ay^2 - 2xy^2 + y^4$

$\therefore x = \dfrac{y^4 - 2ay^2}{2(y^2 - a)}$ $\therefore dx = 2ydy - \dfrac{(y^4 - 2ay^2)ydy}{(y^2 - a)^2}$ . . . . . .

$= ydy + \dfrac{a^2ydy}{(y^2 - a)^2}$ $\therefore du = dy + \dfrac{a^2dy}{(y^2 - a)^2}$, which can be integrated by the method of *indeterminate coefficients* which will be given in art. 55.

PRAXIS.

1. $\int \dfrac{xdx}{\sqrt{a^4 - x^4}} = \frac{1}{2} \sin.^{-1} \dfrac{x^2}{a^2}$.

2. $\int \dfrac{xdx}{\sqrt{x^4 \pm a^4}} = l\sqrt{x^2 + \sqrt{x^4 \pm a^4}}$.

3. $\int \dfrac{xdx}{a^4 + x^4} = \dfrac{1}{2a^2} \tan.^{-1} \dfrac{x^2}{a^2}$.

4. $\int \dfrac{x^{\frac{1}{2}n - 1}dx}{x^n - a^n} = \dfrac{1}{na^{\frac{n}{2}}}\, l\, \dfrac{x^{\frac{n}{2}} - a^{\frac{n}{2}}}{x^{\frac{n}{2}} + a^{\frac{n}{2}}}$.

5. $\int \dfrac{x^{\frac{1}{2}n - 1}dx}{\sqrt{a + bx^n}} = \dfrac{2}{nb^{\frac{1}{2}}}\, l. \left\{ x^{\frac{n}{2}} + \dfrac{\sqrt{a + bx^n}}{b^{\frac{1}{2}}} \right\}$.

6. $\int \frac{x^{\frac{1}{2}n-1}dx}{\sqrt{a-bx^n}} = -\frac{2}{nb^{\frac{1}{2}}}.\cos.^{-1}.\left(\frac{bx^n}{a}\right)^{\frac{1}{2}}$

7. $\int \frac{dx}{x\sqrt{a+bx^n}} = \frac{1}{na^{\frac{1}{2}}}\,l.\frac{(\sqrt{a+bx^n}-\sqrt{a})^2}{bx^n}$ which, when a is negative, contains $\sqrt{-1}$.

8. $\int \frac{dx}{x\sqrt{bx^n-a}} = \frac{2}{na^{\frac{1}{2}}}\sec.^{-1}\left(\frac{bx^n}{a}\right)^{\frac{1}{2}}.$

9. $\int \frac{x^{\frac{1}{2}}dx}{\sqrt{a-bx^3}} = \frac{2}{3b^{\frac{1}{2}}}\sin.^{-1}.\left(\frac{bx^3}{a}\right)^{\frac{1}{2}}.$

10. $\int \frac{x^{\frac{1}{2}}dx}{a^3-5x^3} = \frac{1}{3a\sqrt{5a}}\,l\,\frac{\sqrt{a^3}+\sqrt{5x^3}}{\sqrt{a^3}-\sqrt{5x^3}}.$

11. $\int \frac{axdx+bdx}{(x-c)^2} = al(x-c) - \frac{ac+b}{x-c}.$

12. $\int \frac{axdx-3ax^3dx}{(1+x^2)^3} = \frac{3ax^2+a}{2.(1+x^2)^2}.$

13. $\int \frac{dx}{x^{\frac{1}{2}}\sqrt{a+bx}} = -\frac{2}{b^{\frac{1}{2}}}\sin.^{-1}\sqrt{1-\frac{bx}{a}}.$

14. $\int \frac{2adx}{x\sqrt{a^3+x^3}} = \frac{2}{3a^{\frac{1}{2}}}\,l.\frac{(\sqrt{a^3+x^3}-a^{\frac{3}{2}})^2}{x^3}.$

15. $\int \frac{dx}{x\sqrt{1+x^{\frac{1}{2}}}} = 4l.\frac{\sqrt{1+x^{\frac{1}{2}}}-1}{x^{\frac{1}{2}}}.$

16. $\int \frac{dx\sqrt{1-x^2}}{(1+x)^2} = -2\sqrt{\frac{1-x}{1+x}} - \text{su.vs.}^{-1}x.$

48. *When the variables without the radical are principally in the denominator, we should substitute for some function of their reciprocals.*

Ex. 1. $du = \frac{x^{-p}dx}{ax^2\pm bx+c}.$

Substitute $y = \frac{1}{x}$ $\therefore du = \frac{y^p dy}{a\pm by+cy^2}$, which is of the same form as 47. Ex. 9.

Ex. 2. $du = \dfrac{dx}{x.(a^2+x^2)}$.

Substitute $y = \dfrac{1}{x}$ $\therefore du = \dfrac{-dy}{y.\left(a^2+\dfrac{1}{y^2}\right)} = \dfrac{-ydy}{a^2y^2+1}$

$\therefore u = -\dfrac{1}{2a^2}l(a^2y^2+1) = -\dfrac{1}{2a^2}.l\dfrac{a^2+x^2}{x^2} = \dfrac{1}{a^2}.l\dfrac{x}{\sqrt{a^2+x^2}}$.

Otherwise. $\dfrac{dx}{x} = \dfrac{dx.(a^2+x^2)}{x.(a^2+x^2)} = a^2du + \dfrac{xdx}{a^2+x^2}$ $\therefore$ &c.

Ex. 3. $du = \dfrac{dx}{x^2.(a^2+x^2)}$.

Substitute $y = \dfrac{1}{x}$ $\therefore$ $-dy = \dfrac{dx}{x^2}$ $\therefore du = \dfrac{-dy}{a^2+\dfrac{1}{y^2}}$

$= \dfrac{-y^2dy}{a^2y^2+1} =$, by division, $\dfrac{-dy}{a^2} + \dfrac{dy}{a^2(a^2y^2+1)} = -\dfrac{dy}{a^2}$

$+ \dfrac{\dfrac{1}{a^4}dy}{y^2+\dfrac{1}{a^2}}$ $\therefore u = \dfrac{-y}{a^2} + \dfrac{1}{a^3}\tan.^{-1}ay$

$$= \frac{-1}{a^2x} + \frac{1}{a^3}\tan.^{-1}\frac{a}{x}.$$

Otherwise. $\dfrac{dx}{x^2} = \dfrac{dx.(a^2+x^2)}{x^2.(a^2+x^2)} = a^2du + \dfrac{dx}{a^2+x^2}$ $\therefore$ &c.

Ex. 4. $du = \dfrac{a^3dx}{x^2\sqrt{a^2+x^2}}$.

Substitute $y = \dfrac{a^2}{x}$ $\therefore du = \dfrac{-ydy}{\sqrt{a^2+y^2}}$ $\therefore u = -\sqrt{a^2+y^2}$

$= \dfrac{-a\sqrt{a^2+x^2}}{x}$.

Ex. 5. $du = \dfrac{xdx}{(a+x)^3\sqrt{a^2+ax+x^2}}$.

Substitute $y = \frac{a^2}{a+x}$ $\therefore du = \frac{y^2dy - aydy}{a^4\sqrt{a^2 - ay + y^2}}$, which may be reduced as Art. 47. Ex. 7.

Ex. 6. $du = \frac{(1-x^2)\,dx}{(1+x^2)\sqrt{1+ax^2+x^4}}$.

$$= \frac{(1-x^2)dx}{x.(1+x^2)\sqrt{\frac{1}{x^2}+a+x^2}}$$

Substitute $y = \frac{1}{x} + x \therefore dy = -dx\left\{\frac{1}{x^2} - 1\right\}$;

$y = \frac{1+x^2}{x}$; also $y^2 - 2 = \frac{1}{x^2} + x^2$; hence we have u

$= \int \frac{-dy}{y\sqrt{y^2+a-2}} = (17 \text{ form } 6)\frac{1}{\sqrt{a-2}}$ . . . .

$l. \frac{y}{\sqrt{y^2+a-2} - \sqrt{a-2}} = \frac{1}{\sqrt{a-2}}\, l. \frac{\frac{1}{x}+x}{\sqrt{\frac{1}{x^2}+a+x^2} - \sqrt{a-2}}$.

When a is less than 2, $u = \frac{-1}{2-a}$ arc. rad. $= \sqrt{2-a}$.

sec. $= y = -\frac{1}{\sqrt{2-a}}$ sec.$^{-1}\frac{y}{\sqrt{2-a}} = -\frac{1}{\sqrt{2-a}}$ . . . . . .

sec.$^{-1}$ $\frac{1+x^2}{x\sqrt{2-a}}$.

PRAXIS.

1. $\int \frac{dx}{x\sqrt{ax+x^2}} = -\frac{2}{a}\sqrt{\frac{a+x}{x}}$.

2. $\int \frac{dx}{(1+x)(1-x)^{\frac{2}{3}}} = (1-x)^{-\frac{1}{2}} + \frac{1}{2}\, l \frac{2-\sqrt{1-x}}{\sqrt{1+x}}$.

3. $\int \frac{dx}{x.(a+x)^3} = \left\{\frac{3a+2x}{2a^2(a+x)^2} - \frac{1}{a^3}\, l. \frac{a+x}{x}\right\}$

4. $\int \frac{dx}{x^3(a+x)^2}$

$$= -\frac{1}{a^4}\left\{\frac{1}{2}\left(\frac{a+x}{x}\right)^2 - 3.\frac{a+x}{x} + 3\,l\frac{a+x}{x} + \frac{x}{a+x}\right\}.$$

5. $\int \frac{dx}{(x-a)^2\sqrt{x^2-b^2}}$

6. $\int \frac{dx}{x\sqrt{a^2-ax+x^2}}$

(5 and 6) may be reduced to the form 47 *Ex.* 7.

7. $\int \frac{dx}{x.(a+x)^{\frac{3}{2}}} = \frac{2}{a(a+x)^{\frac{1}{2}}} + \frac{2}{a^{\frac{3}{2}}}\,l\,\frac{(a+x)^{\frac{1}{2}}-a^{\frac{1}{2}}}{x^{\frac{1}{2}}}.$

8. $\int \frac{dx}{x^2\sqrt{a-x}} = -\frac{1}{2a^{\frac{3}{2}}}\,l\left(\frac{2a-x}{2ax} + \frac{1}{x}\sqrt{\frac{a-x}{a}}\right)$

49. *If more radicals or factors than one enter into the expression, substitute for that radical which will render the form the simplest possible.*

Ex. 1. $du = \frac{xdx}{(1+x)^{\frac{1}{3}}-(1+x)^{\frac{1}{2}}}.$

Substitute $y^6=1+x \therefore x=y^6-1$, and $dx=6y^5dy \therefore xdx =6y^5dy\,(y^6-1)$, and $(1+x)^{\frac{1}{3}}-(1+x)^{\frac{1}{2}} = y^2-y^3 \therefore du = -6y^3dy.\frac{y^6-1}{y-1}=$ (by division) $-6y^3dy\left\{y^5+y^4+y^3+y^2+1\right\} \therefore$ &c.

Ex. 2. $du = \frac{xdx\sqrt{b^2+x^2}}{\sqrt{c^2-x^2}}.$

Substitute $y=\sqrt{c^2-x^2} \therefore x^2=c^2-y^2$ and $xdx=-ydy \therefore du=-dy\sqrt{a^2-y^2}$, where $a^2=b^2+c^2$, which will be shown to be a circular area form, Ch. 9.

Ex. 3. $du = \frac{x^{n-1}dx}{(1-x)^n.\,(2x^n-1)^{\frac{1}{2n}}}.$

Substitute $y=(2x^n-1)^{\frac{1}{2n}}$ $\therefore y^{2n}=2x^n-1$ $\therefore y^{2n-1}\,dy=x^{n-1}dx$; also $x^n=\dfrac{y^{2n}+1}{2}$, and $1-x^n=\dfrac{1-y^{2n}}{2}$ $\therefore du=$

$$\frac{y^{2n-1}\,dy}{y.\dfrac{1-y^{2n}}{2}}=\frac{2y^{2n-2}dy}{1-y^{2n}}.$$

Ex. 4. $du=\dfrac{dx}{(1-x)^n.(2x^n-1)^{\frac{1}{2n}}}$.

Substitute $y=\dfrac{(2x^n-1)^{\frac{1}{2n}}}{x}$ and there results $du=\dfrac{y^{2n-2}dy}{1-y^{2n}}$.

Ex. 5. $du=\dfrac{dx}{(x-1)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}}$.

Substitute $y=(x-1)^{\frac{3}{2}}$ $\therefore y^{\frac{2}{3}}=x-1$ $\therefore x+1=y^{\frac{2}{3}}+2$ $\therefore$

$$du=\frac{\frac{2}{3}y^{-\frac{1}{3}}dy}{y.(y^{\frac{2}{3}}+2)^{\frac{1}{2}}}=\tfrac{2}{3}y^{-\frac{5}{3}}dy\left(1+2y^{-\frac{2}{3}}\right)^{-\frac{1}{2}}\ \therefore$$

$$u=-\left(1+2y^{-\frac{2}{3}}\right)^{\frac{1}{2}}=-\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}.$$

PRAXIS.

1. $\displaystyle\int\frac{a^{\frac{1}{2}}+x^{\frac{1}{2}}}{x^{\frac{1}{2}}+x^{\frac{2}{3}}}dx.$

2. $\displaystyle\int\frac{x^{2n-1}dx}{(x^n+a^n)\,(x^n+b^n)^{\frac{1}{2}}}$

$$=\frac{2}{n}\sqrt{x^n+b^n}-\frac{2a^n}{n\sqrt{a^n-b^n}}\tan.^{-1}\sqrt{\frac{x^n+b^n}{a^n-b^n}}.$$

If a be less than b, the fluent is a logarithm.

50. *If two radicals or factors enter into the expression, and the variable in each has the same exponent, and is of the form of the fluent of the part without, it is sometimes useful to substitute for the quotient of the factors.*

Ex. 1. $du = \dfrac{dx}{(x-a)^2(x-b)^3}$.

Substitute $y = \dfrac{x-a}{x-b} \therefore xy - by = x - a$ and $x = \dfrac{by-a}{y-1}$

$\therefore x - a = \dfrac{(b-a)y}{y-1}$ and $x - b = \dfrac{b-a}{y-1}$; also $dx = \dfrac{bdy}{y-1}$

$-\dfrac{(by-a)dy}{(y-1)^2} = -\dfrac{(b-a)dy}{(y-1)^2} \therefore du = \dfrac{-(y-1)^3dy}{(b-a)^4.y^2}$ which can be expanded and integrated.

Ex. 2. $du = \dfrac{x^2dx}{(a^3+x^3)^{\frac{2}{3}}.(b^3-x^3)^{\frac{4}{3}}}$.

Substitute $y = \dfrac{a^3+x^3}{b^3-x^3} \therefore x^3 = \dfrac{b^3y-a^3}{1+y} \therefore$

$x^2dx = \dfrac{\frac{1}{3}(a^3+b^3)dy}{(1+y)^2}$ and $(a^3+x^3)^{\frac{2}{3}}.(b^3-x^3)^{\frac{4}{3}}$ ....

$= \left(\dfrac{a^3+x^3}{b^3-x^3}\right)^{\frac{2}{3}} \times (b^3-x^3)^2 = y^{\frac{2}{3}}\left\{b^3 - \dfrac{b^3y-a^3}{1+y}\right\}^2$ ......

$= \dfrac{(a^3+b^3)^2y^{\frac{2}{3}}}{(1+y)^2} \therefore du = \dfrac{y^{-\frac{2}{3}}dy}{3(a^3+b^3)} \therefore u = \dfrac{y^{\frac{1}{3}}}{a^3+b^3}$

$= \dfrac{1}{a^3+b}.\sqrt[3]{\dfrac{a^3+x^3}{b^3-x^3}}$.

Ex. 3. $du = \dfrac{dx}{(1-ax^2)\sqrt{1-x^2}}$.

Let $y = \sqrt{\dfrac{1-x^2}{1-ax^2}} \therefore y^2 - ay^2x^2 = 1 - x^2 \therefore x^2 = \dfrac{y^2-1}{ay^2-1}$;

$\therefore 1 - x^2 = \dfrac{(a-1)y^2}{ay^2-1}$ and $1 - ax^2 = \dfrac{a-1}{ay^2-1}$. Also

$x = \sqrt{\dfrac{y^2-1}{ay^2-1}} \therefore dx = \sqrt{\dfrac{ay^2-1}{y^2-1}}\left\{\dfrac{ydy}{ay^2-1} - \dfrac{(y^2-1)aydy}{(ay^2-1)^2}\right\}$

$= \sqrt{\dfrac{ay^2-1}{y^2-1}} \cdot \dfrac{(a-1)\,ydy}{(ay^2-1)^2} \therefore du = \dfrac{(a-1)ydy}{\sqrt{y^2-1}\,(ay^2-1)^{\frac{3}{2}}}$ ...

$\div \dfrac{a-1}{ay^2-1}\sqrt{\dfrac{(a-1)y^2}{ay^2-1}} = \dfrac{1}{\sqrt{a-1}} \cdot \dfrac{dy}{\sqrt{y^2-1}}$, which is a lo-

garithmick or a circular form according as a is greater or less than 1.

Ex. 4. $du = \dfrac{xdx}{(a^4-x^4)^{\frac{1}{2}}.(a^2-x^2)} = \dfrac{xdx}{(a^2+x^2)^{\frac{1}{2}}.(a^2-x^2)^{\frac{3}{2}}}$.

Substitute $y = \left(\dfrac{a^2+x^2}{a^2-x^2}\right)^{\frac{1}{2}} \therefore a^2y^2 - x^2y^2 = a^2 + x^2 \therefore$

$x^2 = \dfrac{a^2.(y^2-1)}{y^2+1} \therefore a^2+x^2 = \dfrac{2a^2y^2}{y^2+1}$ and $a^2-x^2 = \dfrac{2a^2}{y^2+1}$; and

$xdx = a^2.\left\{\dfrac{ydy}{y^2+1} - \dfrac{(y^2-1)ydy}{y^2+1)^2}\right\} = \dfrac{2a^2ydy}{(y^2+1)^2} \therefore$

$du = \dfrac{2a^2ydy}{(y^2+1)^2}.\left(\dfrac{y^2+1}{2a^2y^2}\right)^{\frac{1}{2}}.\left(\dfrac{y^2+1}{2a^2}\right)^{\frac{3}{2}} = \dfrac{dy}{2a^2} \therefore$

$u = \dfrac{1}{2a^2}.\left(\dfrac{a^2+x^2}{a^2-x^2}\right)^{\frac{1}{2}}$.

Ex. 5. $du = \dfrac{dx(1+x^2)}{(1-x^2)\sqrt{1+x^4}}$.

Substitute $y = \dfrac{1+x^2}{1-x^2} \therefore y - yx^2 = 1 + x^2 \therefore x^2 = \dfrac{y-1}{y+1}$

and $1+x^4 = 1 + \dfrac{(y-1)^2}{(y+1)^2} = 2.\dfrac{(y^2+1)}{(y+1)^2}$.

Also $x = \left(\dfrac{y-1}{y+1}\right)^{\frac{1}{2}} \therefore dx = \frac{1}{2}\left(\dfrac{y+1}{y-1}\right)^{\frac{1}{2}}.\dfrac{2dy}{(y+1)^2}$ . . . .

$= \dfrac{dy}{(y-1)^{\frac{1}{2}}.(y+1)^{\frac{3}{2}}} \therefore du = \dfrac{ydy}{2^{\frac{1}{2}}.(y^4-1)^{\frac{1}{2}}} = \dfrac{1}{2^{\frac{3}{2}}}.\dfrac{2ydy}{(y^4-1)^{\frac{1}{2}}} \therefore$

$u = \dfrac{1}{2^{\frac{3}{2}}} l\,(y^2 + \sqrt{y^4-1}) =$ &c.

PRAXIS.

1. $\displaystyle\int\frac{dx}{(x-a)^2\,(x-b)} = -\frac{1}{(a-b)^2}.\left\{\frac{x-b}{x-a} + l\frac{x-a}{x-b}\right\}$.

2. $\displaystyle\int\frac{dx}{(x-1)^{\frac{3}{2}}.(x+1)^{\frac{1}{2}}} = -\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}$

3. $\displaystyle\int dx\,\frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{3}{2}}} = 2\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} - 2\tan.^{-1}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$

$$4. \int \frac{dx}{(1-x)^2\sqrt{1+x^2}} = \frac{\sqrt{1+x^2}}{2(1-x)} + \frac{1}{2^{\frac{3}{2}}} l. \frac{1+x+\sqrt{2.(1+x^2)}}{1-x}.$$

Examples 2, 3, 4, 7, 8, of *Praxis* 48 may be integrated by this method.

51. *The following examples are to be integrated by particular artifices of calculation.*

Ex. 1. $du = \dfrac{x^{\frac{1}{2}}dx}{\sqrt{2a-x}} = \dfrac{xdx}{\sqrt{2ax-x^2}}.$

$$du = \frac{adx}{\sqrt{2ax-x^2}} - \frac{adx - xdx}{\sqrt{2ax-x^2}} \therefore u = a\,vs.^{-1}\frac{x}{a} - \sqrt{2ax-x^2}.$$

Ex. 2. $du = dx\sqrt{a^2+x^2}.$

Multiply and divide by $x\sqrt{a^2+x^2}$ $\therefore$

$$du = \frac{a^2xdx + x^3dx}{\sqrt{a^2x^2+x^4}} = \frac{\frac{1}{2}(a^2xdx + 2x^3dx)}{\sqrt{a^2x^2+x^4}} + \frac{\frac{1}{2}a^2xdx}{\sqrt{a^2x^2+x^4}}$$

$$\therefore u = \tfrac{1}{2}(a^2x^2+x^4)^{\frac{1}{2}} + \frac{a^2}{2} l(x + \sqrt{a^2+x^2}).$$

Ex. 3. $du = \dfrac{dx}{x}\sqrt{a^2+x^2}.$

Multiply the numerator and the denominator by $\sqrt{a^2+x^2}$,

$$\therefore du = \frac{x^2dx}{x\sqrt{a^2+x^2}} + \frac{a^2dx}{x\sqrt{a^2+x^2}} = \frac{xdx}{\sqrt{a^2+x^2}} + \frac{a.x^{-2}dx}{\sqrt{a^{-2}+x^{-2}}}$$

$$\therefore (2.17\,(7))\; u = \sqrt{a^2+x^2} + a\,l.\frac{x}{1+\sqrt{a^{-2}x^2+1}}$$

$$= \sqrt{a^2+x^2} + a\,l.\frac{ax}{a+\sqrt{a^2+x^2}}.$$

Ex. 4. $du = \dfrac{dx}{x^2}.\ \sqrt{a^2+x^2}.$

$$\therefore du = \frac{dx}{\sqrt{a^2+x^2}} + \frac{a^2dx}{x^2\sqrt{a^2+x^2}}$$

$$= \frac{dx}{\sqrt{a^2+x^2}} + a^2x^{-3}dx(a^2x^{-2}+1)^{-\frac{1}{2}}$$

$$\therefore u = l.\,(x+\sqrt{a^2+x^2}) - (a^2x^{-2}+1)^{\frac{1}{2}}$$

$$= l.\,(x+\sqrt{a^2+x^2}) - \frac{\sqrt{a^2+x^2}}{x}.$$

$u = \int \frac{dx}{x^2}\sqrt{a^2-x^2}$ contains a circular form.

PRAXIS.

1. $\int \frac{(a+bx)dx}{a^2+x^2} = \tan.^{-1}\frac{x}{a} + \frac{b}{2}\,l\,(a^2+x^2).$

2. $\int \frac{(a+\sqrt{a^2-x^2})dx}{x} = a\,l\,(a-\sqrt{a^2-x^2}) + \sqrt{a^2-x^2}.$

3. $\int \frac{dx}{\sqrt{1+x^2}-\sqrt{1-x^2}} = -\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{2x}$ . . . .

$+\,l.\,\sqrt{x+\sqrt{1+x^2}} - \frac{1}{2}\sin.^{-1}x.$

52. *Integration by parts, or the method of Continuation.*

Since $d.xy = ydx + xdy$, therefore $\int ydx = xy - \int xdy$. In the same manner, since $d.\frac{x}{y} = \frac{dx}{y} - \frac{xdy}{y^2}$, therefore $\int \frac{xdy}{y^2} = \int \frac{dx}{y} - \frac{x}{y}$.

These formulæ, particularly the first, frequently enable us to reduce the fluxion by successive integrations to a simpler form; hence this method is sometimes called the method of *Continuation* or of *Reduction*.

A more general formula of reduction is $\int \mathrm{X}\mathrm{Y}dx = \mathrm{X}\int \mathrm{Y}dx - \int d\mathrm{X}\int \mathrm{Y}dx$, where X and Y are any functions whatever of x.

53. In addition to the elementary forms which have been already investigated, Articles 17 and 18, we shall assume those which will be demonstrated in Ch. 9; viz. that

(1.) $\int dx\sqrt{2ax-x^2}$ = cir. area, rad. $= a$, $v.s. = x$.

(2.) $\int dx\sqrt{a^2-x^2}$ = cir. area, rad. $= a$, distance from centre $= x$.

(3.) $\int dx\sqrt{2ax+x^2}$ = rectangular hyperbolick area, $\frac{1}{2}$ axis $= a$ and abscissa $= x$.

(4.) $\int dx\sqrt{x^2-a^2}$ = the same area, if the abscissa is reckoned from the centre.

54. *Ex.* 1. $du = \dfrac{x^2dx}{\sqrt{x^2 \pm a^2}}$.

Let $du = x.\dfrac{xdx}{\sqrt{x^2+a^2}}$ $\therefore u = x.\sqrt{x^2+a^2} - \int dx\sqrt{x^2+a^2}$

$$= x\sqrt{x^2+a^2} - \int\frac{dx(x^2+a^2)}{\sqrt{x^2+a^2}}$$

$$= x\sqrt{x^2+a^2} - u - \int\frac{a^2dx}{\sqrt{x^2+a^2}},$$

which reduces the integration of $\dfrac{x^2dx}{\sqrt{x^2+a^2}}$ to that of $\int\dfrac{dx}{\sqrt{x^2+a^2}}$, which is a known form.

Let $\int\dfrac{dx}{\sqrt{x^2+a^2}} = \text{A}$, and we have

$2u = x\sqrt{x^2+a^2} - a^2\text{A}$, and taking a^2 either positive or negative, $\int\dfrac{x^2dx}{\sqrt{x^2\pm a^2}} = \frac{1}{2}x\sqrt{x^2\pm a^2} \mp \dfrac{a^2}{2}\text{A}$, where $\text{A} = \int\dfrac{dx}{\sqrt{x^2\pm a^2}}$.

Let the result be $\int\dfrac{x^2dx}{\sqrt{x^2\pm a^2}} = \text{B}$.

Ex. 2. $du = \dfrac{x^2dx}{\sqrt{a^2-x^2}}$.

$du = x.\dfrac{xdx}{\sqrt{a^2-x^2}}$ $\therefore u = -x\sqrt{a^2-x^2} + \int dx\sqrt{a^2-x^2}$,

which is a circular area form (Ch. 9): or it may be reduced to a circular arc form; for

$$u = -x\sqrt{a^2-x^2} + \int\frac{a^2dx - x^2dx}{\sqrt{a^2-x^2}} \therefore$$

$$2u = -x\sqrt{a^2-x^2} + \int\frac{a^2dx}{\sqrt{a^2-x^2}}$$

and $u = -\frac{1}{2} x\sqrt{a^2-x^2} + \frac{a^2}{2}\text{A}$, where $\text{A} = \sin.^{-1}\frac{x}{a}$.

Let the result be $\int \frac{x^2 dx}{\sqrt{a^2-x^2}} = \text{B}$, so that the general form may be $\int \frac{x^2 dx}{\sqrt{x^2 \pm a^2}} = \text{B}$.

Ex. 3. $du = \frac{x^3 dx}{\sqrt{x^2 \pm a^2}}$.

$$du = x^2 . \frac{x dx}{\sqrt{x^2+a^2}} \therefore u = x^2\sqrt{x^2+a^2} - \int 2x dx\sqrt{x^2+a^2}$$

$$= x^2\sqrt{x^2+a^2} - \tfrac{2}{3}(x^2+a^2)^{\frac{3}{2}}$$

$$= \sqrt{x^2+a^2} . \frac{x^2 - 2a^2}{3}.$$

Or $\int \frac{x^3 dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2} . \frac{x^2 \mp 2a^2}{3}$.

Ex. 4. $du = \frac{x^3 dx}{\sqrt{a^2 - x^2}}$

$$u = -x^2\sqrt{a^2-x^2} + \int 2x dx\sqrt{a^2-x^2}$$

$$= -\sqrt{a^2-x^2} . \frac{x^2+2a^2}{3}.$$

Let $\int \frac{x^3 dx}{\sqrt{x^2 \pm a^2}} = \text{C}$.

Ex. 5. $du = \frac{x^4 dx}{\sqrt{x^2 \pm a^2}}$.

$$u = x^3\sqrt{x^2+a^2} - \int 3x^2 dx\sqrt{x^2+a^2}$$

$$= x^3\sqrt{x^2+a^2} - 3u - \int \frac{3a^2x^2 dx}{\sqrt{x^2+a^2}}$$

$$\therefore 4u = x^3\sqrt{x^2+a^2} - 3a^2\text{B}.$$

$$\therefore u = \frac{x^3\sqrt{x^2+a^2}}{4} - \frac{3a^2}{4}\left(\frac{x\sqrt{x^2+a^2}}{2} - \frac{a^2}{2}\text{A}\right)$$

$$= \sqrt{x^2+a^2}\left(\frac{x^3}{4} - \frac{3a^2x}{8}\right) + \frac{3a^4}{8}\text{A}$$

or $\int \dfrac{x^4dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}\left(\dfrac{x^3}{4} \mp \dfrac{3a^2x}{8}\right) + \dfrac{3a^4}{8}\text{A}.$

Ex. 6. $du = \dfrac{x^4dx}{\sqrt{a^2 - x^2}}.$

$$u = -x^3\sqrt{a^2-x^2} + \int 3x^2dx\,\sqrt{a^2-x^2}$$

$$= -x^3\sqrt{a^2-x^2} + 3a^2\int \frac{x^2dx}{\sqrt{a^2-x^2}} - 3u$$

$$\therefore u = -\tfrac{1}{4}x^3\sqrt{a^2-x^2} + \frac{3a^2}{4}\text{B}$$

$$= -\tfrac{1}{4}x^3\sqrt{a^2-x^2} + \frac{3a^2}{4}\left(-\tfrac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\text{A}\right)$$

$$= -\sqrt{a^2-x^2}\left(\frac{x^3}{4} + \frac{3a^2x}{8}\right) + \frac{3a^4}{8}\text{A}.$$

Let $\int \dfrac{x^4dx}{\sqrt{x^2 \pm a^2}} = \text{D}.$

Similarly $\dfrac{x^5dx}{\sqrt{x^2 \pm a^2}}, \quad \dfrac{x^6dx}{\sqrt{x^2 \pm a^2}} \dots \dfrac{x^pdx}{\sqrt{x^2 \pm a^2}}$, where p is a positive integer may be integrated by continued reduction.

55. *Ex.* 1. $du = dx\sqrt{x^2 \pm a^2}.$

Multiply and divide by $\sqrt{x^2 + a^2}$, and we have

$du = \dfrac{x^2dx}{\sqrt{x^2+a^2}} + \dfrac{a^2dx}{\sqrt{x^2+a^2}}$; hence by the preceding article (Ex. 1.)

$$u = \tfrac{1}{2}x\sqrt{x^2+a^2} - \frac{a^2}{2}\text{A} + a^2\text{A}$$

$$= \tfrac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2\text{A}}{2}.$$

Also $\int dx\,\sqrt{a^2-x^2}$ is a known form.

Let $\int dx\,\sqrt{x^2 \pm a^2} = \text{A}'$

Ex. 2. $du = x^2dx\,\sqrt{x^2 \pm a^2}.$

$$u = \frac{x.(x^2+a^2)^{\frac{3}{2}}}{3} - \int \frac{(x^2+a^2)^{\frac{3}{2}}}{3}dx$$

$$= \frac{x(x^2 + a^2)^{\frac{3}{2}}}{3} - \frac{u}{3} - \frac{a^2 A'}{3}$$

$$\therefore u = \frac{x.(x^2 + a^2)^{\frac{3}{2}}}{4} - \frac{a^2 A'}{4}.$$

Or $\int x^2 dx \sqrt{x^2 \pm a^2} = \frac{x.(x^2 \pm a^2)^{\frac{3}{2}}}{4} \mp \frac{a^2 A'}{4}$.

Ex. 3. $du = x^2 dx \sqrt{a^2 - x^2}$.

$$u = -\frac{x.(a^2 - x^2)^{\frac{3}{2}}}{3} + \int \frac{(a^2 - x^2)^{\frac{3}{2}}}{3} dx$$

$$= -\frac{x(a^2 - x^2)^{\frac{3}{2}}}{3} + \frac{a^2 A'}{3} - \frac{u}{3}$$

$$\therefore u = -\frac{x(a^2 - x^2)^{\frac{3}{2}}}{4} + \frac{a^2 A'}{4}.$$

Let $\int x^2 dx \sqrt{x^2 \pm a^2} = B'$

Ex. 4. $du = x^3 dx \sqrt{x^2 \pm a^2}$.

$$u = \frac{x^2.(x^2 + a^2)^{\frac{3}{2}}}{3} - \int \frac{(x^2 + a^2)^{\frac{3}{2}}}{3} \cdot 2x dx$$

$$= \frac{x^2(x^2 + a^2)^{\frac{3}{2}}}{3} - \tfrac{2}{3} \cdot \frac{(x^2 + a^2)^{\frac{5}{2}}}{5}$$

$$= (x^2 + a^2)^{\frac{3}{2}} \left(\frac{x^2}{5} - \frac{2a^2}{15} \right).$$

Or $\int x^3 dx \sqrt{x^2 \pm a^2} = (x^2 \pm a^2)^{\frac{3}{2}} \left(\frac{x^2}{5} \mp \frac{2a^2}{15} \right)$.

Ex. 5. $du = x^3 dx \sqrt{a^2 - x^2}$.

$$u = -\frac{x^2.(a^2 - x^2)^{\frac{3}{2}}}{3} + \tfrac{2}{3} \int (a^2 - x^2)^{\frac{3}{2}} x dx$$

$$= -\frac{x^2(a^2 - x^2)^{\frac{3}{2}}}{3} - \tfrac{2}{3} \frac{(a^2 - x^2)^{\frac{5}{2}}}{5}$$

$$= -(a^2 - x^2)^{\frac{3}{2}} \left(\frac{x^2}{5} + \frac{2a^2}{15} \right).$$

Let $\int x^3dx\sqrt{x^2 \pm a^2} = \text{C}$.

Ex. 6. $du = x^4dx\sqrt{x^2 \pm a^2}$.

$$u = \frac{x^3.(x^2 + a^2)^{\frac{3}{2}}}{3} - \int (x^2 + a^2)^{\frac{3}{2}} x^2dx$$

$$u = \frac{x^3.(x^2 + a^2)^{\frac{3}{2}}}{3} - u - a^2\text{B}'$$

$$\therefore 2u = \frac{x^3(x^2 + a^2)^{\frac{3}{2}}}{3} - a^2\left(\frac{x(x^2 + a^2)^{\frac{3}{2}}}{4} - \frac{a^2\text{A}'}{4}\right)$$

$$\therefore\ u = (x^2 + a^2)^{\frac{3}{2}}\left(\frac{x^3}{6} - \frac{a^2x}{8}\right) + \frac{a^4}{8}\text{A}'.$$

Or $\int x^4dx\sqrt{x^2 \pm a^2} = (x^2 \pm a^2)^{\frac{3}{2}}\left(\frac{x^3}{6} \mp \frac{a^2x}{8}\right) + \frac{a^4}{8}\text{A}'.$

Ex. 7. $du = x^4dx\sqrt{a^2 - x^2}$,

It may be shown that

$$u = -\frac{x^3.(a^2 - x^2)^{\frac{3}{2}}}{6} + \frac{a^2}{2}\text{B}'$$

$$= -\frac{x^3(a^2 - x^2)^{\frac{3}{2}}}{6} + \frac{a^2}{2}\left(-\frac{x(a^2 - x^2)^{\frac{3}{2}}}{4} + \frac{a^2\text{A}'}{4}\right)$$

$$= -(a^2 - x^2)^{\frac{3}{2}}\left(\frac{x^3}{6} + \frac{a^2x}{8}\right) + \frac{a^4\text{A}'}{8}.$$

And thus $\int x^pdx\sqrt{x^2 \pm a^2}$ may be reduced in the same manner as $\int \frac{x^pdx}{\sqrt{x^2 \pm a^2}}$, provided that p is a positive integer.

The results A, B, C, &c., and A$'$, B$'$, C$'$, &c. show that when the index p is odd, the fluent may be obtained in Algebraick terms; but that when it is even, the integration depends upon the quadrature of the circle or of the hyperbola.

When p is odd the forms are integrable by the method of Art. 46.

56. *All fluents of the forms* $\int \frac{x^pdx}{\sqrt{x^2 \pm 2ax}}$ *and* $\int x^pdx\sqrt{x^2 \pm 2ax}$, *where* p *is a positive integer, may be reduced to the above forms.*

For substitute $y = x \pm a$, and the forms are reduced to $\int \frac{(y \pm a)^p dy}{\sqrt{y^2 \pm a^2}}$ and $\int (y \pm a)^p \sqrt{y^2 \pm a^2}\, dy$, which, since p is a positive integer, will terminate when expanded, and each term may be integrated as in the preceding articles.

57. *Integral Tables.*

Meyer Hirsch, a German mathematician, has published a collection of Tables, in which are registered those Integral Formulæ which occur the most frequently in calculation. This work, which has been lately translated, is doubtless a useful book of reference, but yet the student should not rest satisfied with a knowledge of the methods of Integration, but should be able himself to integrate the Formulæ without referring to the tables.

In these tables, at pages 119 and 130 (English translation) $\int \frac{x^p dx}{\sqrt{x^2 \pm a^2}}$ and $\int x^p dx \sqrt{x^2 \pm a^2}$ are calculated from $p = 1$ to $p = 11$ or 12; and it is obvious that if all of the first form had been alone calculated, they would have enabled us to integrate the latter form, which is the same as

$$\int \frac{x^{p+2}dx}{\sqrt{x^2 \pm a^2}} \pm a^2 \int \frac{x^p dx}{\sqrt{x^2 \pm a^2}}.$$

58. The process of reducing du to the forms ... $d\mathrm{C}$, $d\mathrm{B}$, $d\mathrm{A}$ is, in general, rendered more simple by assuming certain rectangles ... P, Q, R, such that $d\mathrm{P}$ may contain du and $d\mathrm{C}$, $d\mathrm{Q}$ may contain $d\mathrm{C}$ and $d\mathrm{B}$, and $d\mathrm{R}$ may contain $d\mathrm{B}$ and $d\mathrm{A}$.

The rules for the assumption of the rectangles will be given in the second volume; but we shall show the method by applying it to the forms of Articles 54 and 55, when p is a negative integer.

Ex. 1. Required $\int \frac{dx}{x^4 \sqrt{a + bx^2}}$, where a and b may be either positive or negative.

Let	Assume
$d\mathrm{A} = \frac{dx}{\sqrt{a + bx^2}}$	$\mathrm{P} = x^{-3}\sqrt{a+bx^2}$ $\therefore$
$d\mathrm{B} = \frac{dx}{x^2\sqrt{a + bx^2}}$	$d\mathrm{P} = -\frac{3dx\sqrt{a+bx^2}}{x^4} + bd\mathrm{B}$
$d\mathrm{C} = \frac{dx}{x^4\sqrt{a+bx^2}}$	$= -3ad\mathrm{C} - 2bd\mathrm{B}$
	$\therefore \mathrm{C} = -\frac{\mathrm{P}}{3a} - \frac{2b\mathrm{B}}{3a}$.

Assume $Q = x^{-1}\sqrt{a+bx^2}$ $\therefore$

$$dQ = \frac{-dx\sqrt{a+bx^2}}{x^2} + b\,dA$$

$$= -a\,dB \;\therefore\; B = -\frac{Q}{a} = \frac{-\sqrt{a+bx^2}}{ax};\;\therefore$$

$$C = \frac{-\sqrt{a+bx^2}}{3ax^3} + \frac{2b\sqrt{a+bx^2}}{3a^2x} = \sqrt{a+bx^2}\left(-\frac{1}{3ax^3} + \frac{2b}{3a^2x}\right).$$

The value of B might have been obtained from Art. 13.

Ex. 2. Required $\int \frac{dx}{x^5\sqrt{a+bx^2}} = C.$

Let $dA = \frac{dx}{x\sqrt{a+bx^2}}$ $dB = \frac{dx}{x^3\sqrt{a+bx^2}}$	Assume $P = x^{-4}\sqrt{a+bx^2}$ $\therefore$ $dP = \frac{-4dx\sqrt{a+bx^2}}{x^5} + b\,dB$ $= -4a\,dC - 3b\,dB$ $\therefore C = \frac{-P}{4a} - \frac{3bB}{4a}.$

Assume $Q = x^{-2}\sqrt{a+bx^2}$ $\therefore$

$$dQ = \frac{-2dx\sqrt{a+bx^2}}{x^3} + b\,dA$$

$$= -2a\,B - b\,dA \;\therefore\; B = -\frac{Q}{2a} - \frac{bA}{2a}\cdot$$

$$\therefore C = \frac{-\sqrt{a+bx^2}}{4ax^4} + \frac{3b}{8a^2}\cdot\frac{\sqrt{a+bx^2}}{x^2} + \frac{3b^2}{8a^2}A.$$

$$= \sqrt{a+bx^2}\left(-\frac{1}{4ax^4} + \frac{3b}{8a^2x^2}\right) + \frac{3b^2}{8a^2}A,$$

which reduces $\int \frac{dx}{x^5\sqrt{a+bx^2}}$ to $\int \frac{dx}{x\sqrt{a+bx^2}}$, which is a known transcendental form.

Ex. 3. Required $\int x^{-4}dx\sqrt{a+bx^2} = C.$

Let $dA = dx\sqrt{a+bx^2}$ $dB = x^{-2}dx\sqrt{a+bx^2}$	Assume $P = x^{-3}.(a+bx^2)^{\frac{3}{2}}.$ $dP = -\frac{3dx(a+bx^2)^{\frac{3}{2}}}{x^4} + 3b\,dB$ $= -3a\,dC \;\therefore\; C = -\frac{(a+bx^2)^{\frac{3}{2}}}{3ax^3}.$

Ex. 4. Required $\int x^{-5}dx\sqrt{a+bx^2}=u$.

Assume $P=x^{-4}\sqrt{a+bx^2}\ \therefore dP=-4du+\dfrac{bdx}{x^3\sqrt{a+bx^2}}$.

Assume $Q=x^{-2}\sqrt{a+bx^2}\ \therefore dQ=-\dfrac{2dx\sqrt{a+bx^2}}{x^3}+\dfrac{bdx}{x\sqrt{a+bx^2}}$

$$=\frac{-2adx}{x^3\sqrt{a+bx^2}}-\frac{bdx}{x\sqrt{a+bx^2}}$$

$$\therefore \int\frac{dx}{x^3\sqrt{a+bx^2}}=-\frac{\sqrt{a+bx^2}}{2ax^2}-\frac{b}{2a}\int\frac{dx}{x\sqrt{a+bx^2}}$$

$$\therefore u=-\frac{\sqrt{a+bx^2}}{4x^4}-\frac{b\sqrt{a+bx^2}}{8ax^2}-\frac{b^2}{8a}\int\frac{dx}{x\sqrt{a+bx^2}}$$

$$=-\sqrt{a+bx^2}\left(\frac{1}{4x^4}+\frac{b}{8ax^2}\right)-\frac{b^2}{8a}\int\frac{dx}{x\sqrt{a+bx^2}}.$$

By similar assumptions $\int\dfrac{x^p dx}{\sqrt{a+bx^2}}$ and $\int x^p dx\sqrt{a+bx^2}$, where p is an integer, either positive or negative, may always be reduced and integrated.

It may be observed, that by these assumptions p is increased or diminished each time by 2. By proper assumptions, in all cases of a binomial fluent, the index of x without the vinculum may be increased or diminished by its index under the vinculum.

Ex. 5. $u=\int\dfrac{dx}{(x^2+1)^3}$.

Assume $P=x.(x^2+1)^{-2}\ \therefore dP=\dfrac{dx}{(x^2+1)^2}-\dfrac{4x^2dx}{(x^2+1)^3}\ \ldots\ldots$

$$=\frac{dx-3x^2dx}{(x^2+1)^3}=\frac{-3(x^2+1)dx}{(x^2+1)^3}+\frac{4dx}{(x^2+1)^3}\ \ldots\ldots\ldots\ldots$$

$$\therefore u=\frac{P}{4}+\tfrac{3}{4}\int\frac{dx}{(x^2+1)^2}.$$

Assume $Q=x(x^2+1)^{-1}\ \therefore dQ=\dfrac{dx}{x^2+1}-\dfrac{2x^2dx}{(x^2+1)^2}=\dfrac{dx-x^2dx}{(x^2+1)^2}$

$$=\frac{-(x^2+1)dx}{(x^2+1)^2}+\frac{2dx}{(x^2+1)^2}\ \therefore\int\frac{dx}{(x^2+1)^2}=\frac{Q}{2}+\tfrac{1}{2}\tan.^{-1}x.;$$

hence $u=\dfrac{x}{4(x^2+1)^2}+\dfrac{3x}{8(x^2+1)}+\tfrac{3}{8}\tan.^{-1}x.$

PRAXIS.

1. $\int z^2 x^{4-n} dx$ where $dz = (a + bx^n)^m \; dx$.

2. $\int \frac{x dx}{\sqrt{a + x}} = (a+x)^{\frac{1}{2}} \cdot \frac{2x - 4a}{3}$.

3. $\int \frac{x^5 dx}{(a^2 - x^2)^{\frac{3}{2}}} = \frac{8a^4 - 4a^2x^2 - x^4}{3\sqrt{a^2 - x^2}}$.

59. *The method of Indeterminate Coefficients.*

This method is to be used when the fluent is a rational fraction. The denominator of the fraction is to be treated as an equation, and its roots or factors are to be ascertained from the theory of equations; indeterminate coefficients of the factors are then assumed, which are to be determined from the conditions of the question, and the fluent is decomposed into others of a simpler form.

Ex. 1. $du = \frac{2a dx}{a^2 - x^2}$.

$a^2 - x^2 = 0$ when decomposed is $(a+x)\,(a-x) = 0$; assume therefore $\frac{2a}{a^2 - x^2} = \frac{A}{a+x} + \frac{B}{a-x} = \frac{(A+B)a - (A-B)x}{a^2 - x^2}$ $\therefore$ $A + B = 2$ and $A - B = 0$ $\therefore$ $A = 1$ and $B = 1$, or $du = \frac{dx}{a+x} + \frac{dx}{a-x}$, and $u = l\,\frac{a+x}{a-x}$.

Ex. 2. $du = \frac{dx}{x^2 - 5x + 6}$.

$x^2 - 5x + 6 = 0 = (x-3)\,(x-2)$; assume $\frac{1}{x^2 - 5x + 6} = \frac{A}{x-3} + \frac{B}{x-2} = \frac{(A+B)x - (2A+3B)}{x^2 - 5x + 6}$ $\therefore$ $A + B = 0$ and $2A + 3B = -1$ $\therefore$ $B = -1$ and $A = +1$ $\therefore$ $du = \frac{dx}{x-3} - \frac{dx}{x-2}$ and $u = l\frac{x-3}{x-2}$.

Ex. 3. $du = \frac{3x - 5}{x^2 - 6x + 8}\,dx$.

$x^2 - 6x + 8 = 0 = (x-2)\,(x-4)$; assume $\frac{3x-5}{x^2 - 6x + 8} \ldots = \frac{A}{x-2} + \frac{B}{x-4} = (A + B)x - (4A + 2B)$ $\therefore$ $A + B = 3$ and

$4A + 2B = 5$ ∴ by elimination, $B = \frac{7}{2}$ and $A = -\frac{1}{2}$ ∴

$$du = \frac{-\frac{1}{2}dx}{x-2} + \frac{\frac{7}{2}dx}{x-4} \therefore u = 7.\, l\sqrt{x-4} - l\sqrt{x-2}.$$

Ex. 4. $du = \dfrac{a^3+bx^2}{a^2x-x^3}dx.$

$a^2x - x^3 = 0 = x(a-x)(a+x)$; assume $\dfrac{a^3+bx^2}{a^2x-x^3} = \dfrac{A}{x} +$

$\dfrac{B}{a-x} + \dfrac{C}{a+x} = \dfrac{Aa^2 - Ax^2 + Bax + Bx^2 + Cax - Cx^2}{a^2x-x^3}$ ∴ $Aa^2 = a^3$ or $A = a$; $-A + B - C = b$ or $B - C = b + a$; and $Ba + Ca = 0$ or $B + C = 0$, whence $B = \dfrac{b+a}{2}$ and $C = -\dfrac{b+a}{2}$; and by substitution, we have $du = \dfrac{adx}{x} + \dfrac{a+b}{2}\cdot\dfrac{dx}{a-x} - \dfrac{a+b}{2}\cdot\dfrac{dx}{a+x}$

∴ $u = alx - (a+b)\, l\sqrt{a^2-x^2}$.

Ex. 5. $du = \dfrac{xdx}{x^2+4ax-b^2}$.

$x^2 + 4ax - b^2 = 0 = (x + 2a + \sqrt{4a^2+b^2})(x + 2a - \sqrt{4a^2+b^2})$ $=$, by substitution, $(x+K)(x+L)$; assume $\dfrac{x}{x^2+4ax-b^2}$ $= \dfrac{A}{x+K} + \dfrac{B}{x+L} = \dfrac{(A+B)x + AL + BK}{(x+K)(x+L)}$ ∴ $A + B = 1$ and $AL +$ $BK = 0$, ∴ $A = \dfrac{K}{K-L}$ and $B = \dfrac{-L}{K-L}$ ∴ $u = \dfrac{K}{K-L}\, l(x+K)$ $- \dfrac{L}{K-L}\, l(x+L)$, in which K and L are to be replaced by their values, $2a + \sqrt{4a^2+b^2}$ and $2a - \sqrt{4a^2+b^2}$.

It will be shown in the second volume that every rational fraction may be integrated either algebraically or by reducing it to a transcendental form.

60. The assumptions of the preceding article will always enable us to integrate a rational fluent, provided that the roots of the equation are *unequal* and *real*. If the equation contains either equal or imaginary roots, these assumptions fail to answer the purpose, and others must be adopted.

First, suppose all the roots unequal and real, and let $du = \dfrac{(Px^3 + Qx^2 + Rx + S)dx}{(x-a)(x-b)(x-c)(x-d)}$, then if we assume . . .

$\frac{Px^3 + Qx^3 + Rx + S}{(x-a)(x-b)(x-c)(x-d)} = \frac{A}{x-a} + \frac{B}{(x-b)} + \frac{C}{(x-c)} + \frac{D}{x-d}$, and, reducing the fractions to a common denominator, equate like terms, it is manifest that we shall have as many equations as there are indeterminate coefficients, and the fluent may be integrated.

But if we suppose that two of the roots are equal, or that $du = \frac{(Px^3 + Qx^2 + Rx + S)dx}{(x-a)^2(x-c)(x-d)}$, and make the same assumptions as before, we have $\frac{Px^3 + Qx^2 + Rx + S}{(x-a)^2(x-c)(x-d)} = \frac{A+B}{x-a} + \frac{C}{x-c} + \frac{D}{x-d}$, from which we can obtain only three equations, and consequently the coefficients cannot be determined. (Alg. 145.)

If three or more roots are equal, the number of independent equations will be still further diminished. These results show that the proposed fraction cannot be analytically expressed under the required form, and we must look out for some other form that will enable us to effect the integration.

Now when *two* roots are equal, if we assume . . . .
$\frac{Px^3 + Qx^2 + Rx + S}{(x-a)^2(x-c)(x-d)} = \frac{A+Bx}{(x-a)^2} + \frac{C}{x-c} + \frac{D}{x-d}$, and reduce these fractions as before, it is obvious that we shall have four independent equations, and that the coefficients may be determined. But we shall now have to integrate a fluent of the form $\frac{(A+Bx)dx}{(x-a)^2}$: for which purpose assume $\frac{A+Bx}{(x-a)^2} = \frac{A'}{(x-a)^2} + \frac{A''}{x-a} = \frac{A' - A''a + A''x}{(x-a)^2}$, which determines A' and A'' in terms of A and B, and consequently shows that we may assume $du = \left\{\frac{A'}{(x-a)^2} + \frac{A''}{x-a} + \frac{C}{x-c} + \frac{D}{x-d}\right\} dx$, all of which are of the logarithmick form, except the first, which can be integrated by Rule 1.

Next let *three* roots be equal.

Assume $\frac{Px^3 + Qx^2 + Rx + S}{(x-a)^3(x-c)} = \frac{A + Bx + Cx^2}{(x-a)^3} + \frac{D}{x-c}$, from

which, as before, the coefficients may be determined, and we shall have to integrate $\frac{(A + Bx + Cx^2)dx}{(x-a)^3}$.

Assume $\frac{A + Bx + Cx^2}{(x-a)^3} = \frac{A'}{(x-a)^3} + \frac{A''}{(x-a)^2} + \frac{A'''}{x-a}$

$$= \frac{A' - aA'' + a^2A''' + (A'' - 2aA''')x + A'''x^2}{(x-a)^3},$$

from which A', A'', A''' may be found in terms of A, B, and C, and consequently du may be assumed $= \frac{A'}{(x-a)^3} + \frac{A''}{(x-a)^2}$ $+ \frac{A'''}{x-a} + \frac{D}{x-d}$.

It is evident that the same method may be applied to any number of equal roots.

If the equation contains more than one equal root, similar assumptions must be made for each of them.

Thus, if $du = \frac{dx}{(x-1)^2 (x+1)^3}$, assume

$$\frac{1}{(x-1)^2 . (x+1)^3} = \frac{A}{(x-1)^2} + \frac{A'}{x-1} + \frac{B}{(x-1)^3} + \frac{B'}{(x-1)^2} + \frac{B''}{x-1}.$$

We have supposed the exponent of x in the numerator to be less by unity than its exponent in the denominator. If it should be greater, the fluent may always be reduced to this assumed form by division, and the monomials integrated by the preceding rules.

$$\text{Thus } \int \frac{x^3 + bx^2}{x^2 - a^2} du = \int . \left\{ xdx + bdx + \frac{(a^2x + a^2b)dx}{x^2 - a^2} \right\}$$
$$= \frac{x^2}{2} + bx + a^2 \int \frac{(x + b)dx}{x^2 - a^2}.$$

61. Next suppose that the denominator contains *imaginary* roots.

It is shown in the Algebra, Art. 277, that imaginary roots enter equations by pairs. If $\pm \alpha + \beta \sqrt{-1}$ is one root, its conjugate is $\pm \alpha - \beta \sqrt{-1}$, and consequently the product of two conjugate factors takes the form of $x^2 \mp 2\alpha x + \alpha^2 + \beta^2$, which $= (x \mp \alpha)^2 + \beta^2$.

Let $du = \frac{(Px^2 + Qx + R)dx}{(x-a)(x^2 + 2\alpha x + \alpha^2 + \beta^2)}$;

assume $\frac{Px^2 + Qx + R}{(x-a)(x^2 + 2\alpha x + \alpha^2 + \beta^2)} = \frac{A}{x-a} + \frac{Mx + N}{(x + \alpha)^2 + \beta^2}$;

from which we obtain three independent equations for determining A, M, and N, and the integration of du is in consequence reduced to $\int \frac{(Mx+N)dx}{(x+\alpha)^2+\beta^2}$.

But $\frac{(Mx+N)dx}{(x+\alpha)^2+\beta^2} = \frac{(Mx+M\alpha)dx}{(x+\alpha)^2+\beta^2} + \frac{(N-M\alpha)dx}{(x+\alpha)^2+\beta^2}$, which are transcendental forms; whence

$$u = A\,l(x-a) + M\,l\sqrt{(x+\alpha)^2+\beta^2} + \frac{N-M\alpha}{\beta}\tan.^{-1}\frac{x+\alpha}{\beta}.$$

If there are more than one pair of conjugate roots, we must assume $\frac{du}{dx} = \frac{A}{x-a} + \frac{Mx+N}{(x+\alpha)^2+\beta^2} + \frac{M'x+N'}{(x+\gamma)^2+\delta^2} + \&c.$, and having determined the coefficients A, M, M′, &c. N, N′, &c., each term may be integrated as before.

If there are *equal* conjugate roots, and their number be p, we must assume

$$\frac{du}{dx} = \frac{A}{x-a} + \frac{Mx+N}{((x+\alpha)^2+\beta^2)^p} + \frac{M'x+N'}{((x+\alpha)^2+\beta^2)^{p-1}} + \&c.,$$

which reduces the integration of du to one of the form $\int \frac{(Mx+N)dx}{((x+\alpha)^2+\beta^2)^p}$, which by substitution, $= M\int \frac{ydy}{(y^2+\beta^2)^p}$ $+ (N - M\alpha)\int \frac{dy}{(y^2+\beta^2)^p}$, of which the first is a known form, and the second may be reduced by the method of continuation to $\int \frac{dy}{y^2+\beta^2}$, a tangential form.

PRAXIS.

1. $\int \frac{dx}{x(1-x)} = l\,\frac{x}{1-x}$.

2. $\int \frac{dx}{x(a^2-x^2)} = \frac{1}{a^2}\,l\,\frac{x}{\sqrt{a^2-x^2}}$.

3. $\int \frac{dx}{x^3-7x^2+12x} = \frac{1}{12}\,lx + \frac{1}{4}\,l(x-4) - \frac{1}{3}\,l(x-3)$.

4. $\int \frac{dx}{(x-a)(x-b)(x-c)} = \frac{l(x-a)}{(a-b)(a-c)} + \frac{l(x-b)}{(a-b)(c-b)}$ $+ \frac{l(x-c)}{(a-c)(b-c)}$.

5. $\int \frac{x^2dx}{x^2-a^2} = x + \frac{a}{2} l \frac{x-a}{x+a}$.

6. $\int \frac{dx}{x^4-2x^3-x^2+2x} = \frac{1}{2} lx - \frac{1}{6} l(x+1) - \frac{1}{2} l(x-1) + \frac{1}{6} l(x-2)$.

7. $\int \frac{x^2dx}{(x+a)(x-a)^2} = -\frac{a}{2(x-a)} + \frac{3}{4}l(x-a) + \frac{1}{4}l(x+a)$.

8. $\int \frac{dx}{(x^2-1)^2} = -\frac{x}{2(x^2-1)} + \frac{1}{4}l\frac{x+1}{x-1}$.

9. $\int \frac{x^4dx}{x^3+bx^2-a^2x-a^2b} = \frac{x^2}{2} - bx - \frac{b^4l(x+b)}{a^2-b^2} \dots \dots + \frac{a^3l(x+a)}{2(a-b)} + \frac{a^3l(x-a)}{2(a+b)}$.

10. $\int \frac{(a+bx)dx}{x^3-1} = \frac{a+b}{3}l(x-1) - \frac{a+b}{3}l\sqrt{x^2+x+1} \dots \dots + \frac{b-a}{\sqrt{3}}\tan.^{-1}\frac{2x+1}{\sqrt{3}}$.

11. $\int \frac{xdx}{(x+a)(x^2+a^2)} = -\frac{1}{2a}l(x+a) + \frac{1}{4a}l(x^2+a^2) \dots \dots + \frac{1}{2a}\tan.^{-1}\frac{x}{a}$.

12. $\int \frac{(x^4+2x^3+3x^2+3)dx}{(x^2+1)^3} = \frac{7x^3-8x^2+9x-4}{(x^2+1)^3} + \frac{15}{8}\tan.^{-1}x$.

13. $\int \frac{x^5dx}{x^6-x^4+2x^3+5x^2-6x+6} = \frac{90}{7\times 61} l\left(\frac{x^2}{3}+x+1\right) + \frac{22}{61} l\left(\frac{x^2}{2}-x+1\right) - \frac{1}{14} l(x^2-x+1) - \frac{54\sqrt{3}}{7\times 61}\tan.^{-1}\frac{2x+3}{\sqrt{3}} + \frac{92}{61}\tan.^{-1}(x-1) - \frac{4\sqrt{3}}{21}\tan.^{-1}\frac{2x-1}{\sqrt{3}}$.

Leybourn's Math. Rep. vol. i. p. 74.

62. *Integration by Infinite Series.*

If the proposed fluent is not integrable by substitution, as in Art. 29, nor by either of the methods of continuation and of indeterminate coefficients, and if it cannot be reduced to a known form by any artifice of calculation, it may always

be expanded into an infinite series, each term of which may be integrated.

The developement of a function into a series is also of great importance in other branches of the subject. The principal object, in the developement, is to obtain series which will converge with the greatest rapidity. If a large integral value is to be assigned to the variable, we shall require a descending series; if it is to be a small fraction, an ascending series; and in either case we may obtain an approximate value of the fluent.

It may be observed in this place, that when the series does not terminate, the symbol $=$, which connects the function and its developement, does not *necessarily* represent numerical equality; for if the series does not converge, it is the symbol of a quantity of which the mind can form no precise idea, and even if it does converge, it may only represent an approximate value. In these cases, all that the symbol $=$ denotes, is, that the two functions are convertible into each other by division, evolution, or some other analytical process. Vid. Woodhouse's "Principles of Analytical Calculation," p. 3.

63. *Transcendental Fluents.*

These are, in general, to be integrated by the same methods as the algebraick. By transformation they may always be converted into algebraick fluents; but it frequently requires great analytical skill to reduce them to elementary forms.

When the fluent is partly algebraick and partly transcendental, *integration by parts* will almost always be found to be the method which should be adopted.

In all cases, they, as well as algebraick fluents, may be expanded into series, and an approximate value may be obtained.

64. *Exponential and Logarithmick Fluents.*

Ex. 1. $du = x^3a^xdx$.

Integrating by parts,

$$u = x^3.\frac{a^x}{la} - \int\frac{a^x}{la}.3x^2dx,$$

$$= \frac{a^xx^3}{la} - \frac{3}{la}\left\{x^2.\frac{a^x}{la} - \int\frac{a^x}{la}.2xdx\right\},$$

$$=\frac{a^x x^3}{la}-\frac{3a^x x^2}{la^2}+\frac{2.3}{la^2}\left\{\frac{xa^x}{la}-\frac{a^x}{la^2}\right\},$$

$$=\frac{a^x}{la}\left\{x^3-\frac{3x^2}{la}+\frac{2.3x}{la^2}-\frac{2.3}{la^3}\right\}.$$

It is evident that the form $\int x^n a^x dx$, where n is a positive integer, may be reduced and integrated by the same method.

Ex. 2. $du=\frac{a^x dx}{x^3}=x^{-3}a^x dx.$

Let $dA=x^{-1}a^x dx$	Assume $P=x^{-2}a^x$ $\therefore$
$dB=x^{-2}a^x dx$	$dP=-2du+ladB$ $\therefore$ $u=-\frac{P}{2}+\frac{la}{2}.B.$

Assume $Q=x^{-1}a^x$ $\therefore$ $dQ=-dB+ladA$ $\therefore$ $B=-Q+la.A,$

$$\therefore u=\frac{-P}{2}-\frac{la.Q}{2}+\frac{la^2}{2}.A=\frac{-a^x}{2x^2}-\frac{laa^x}{2}+\frac{la^2}{2}\int\frac{a^x dx}{x}.$$

Hence it is manifest that the form $\int x^{-n}a^x dx$, where n is an integer, may be reduced to $\int\frac{a^x dx}{x}$, an elementary form which has hitherto baffled the skill of the most eminent analysts, and which can be integrated only by developement.

Ex. 3. $du=\frac{x^3 dx}{a^x}=x^3 a^{-x}dx$

$$\therefore u=\frac{-x^3a^{-x}}{la}+\int\frac{a^{-x}}{la}.3x^2dx$$

$$=\frac{-x^3a^{-x}}{la}-\frac{3}{la}\left\{\frac{x^2a^{-x}}{la}-\int\frac{a^{-x}}{la}.2xdx\right\}$$

$$=\frac{-x^3a^{-x}}{la}-\frac{3x^2a^{-x}}{la^2}-\frac{3.2}{la^2}\left\{\frac{xa^{-x}}{la}-\int\frac{a^{-x}}{la}dx\right\}$$

$$=\frac{-a^{-x}}{la}\left\{x^3+\frac{3x^2}{la}+\frac{3.2x}{la^2}+\frac{3.2.1}{la^3}\right\}.$$

Ex. 4. $du=\frac{dx}{a^x x^3}.$

Let $dA=a^{-x}dx$	Assume $P=x^{-2}a^{-x}$ $\therefore$ $dP=-2du-ladC$
$dB=x^{-1}a^{-x}dx$	$\therefore$ $u=\frac{P}{2}-\frac{la.C}{2}.$
$dC=x^{-2}a^{-x}dx$	Assume $Q=x^{-1}a^{-x}$ $\therefore$ $dQ=-dC-ladB$

$\therefore\ -\text{C}=\text{Q}-la\,.\,\text{B}$. But a similar assumption will not give B in terms of A; hence we have $u=\dfrac{\text{P}}{2}+\dfrac{la.\text{Q}}{2}+\dfrac{la^2}{2}\text{B}$

$$=\frac{x^{-2}a^{-x}}{2}+\frac{lax^{-1}a^{-x}}{2}+\frac{la^2}{2}.\int\frac{a^{-x}dx}{x}.$$

These four examples might have been integrated by substituting $y=a^x$.

Ex. 5. $du=\dfrac{e^x x dx}{(1+x)^2}$.

Assume $\text{P}=e^x.(1+x)^{-1}\ \therefore d\text{P}=\dfrac{e^x dx}{1+x}-\dfrac{e^x dx}{(1+x)^2}=\dfrac{e^x x dx}{(1+x)^2}=du$

$$\therefore u=\text{P}=\frac{e^x}{1+x}.$$

Ex. 6. $du=e^{x^{\frac{1}{2}}}xdx$.

Substitute $y^2=x\ \therefore y^4=x^2$ and $2y^3dy=xdx\ \therefore du=2e^y y^3 dy$

$\therefore$ (Ex. 1.) $u=2e^y\{y^3-3y^2+2.3y-2.3\}$

$$=2e^{x^{\frac{1}{2}}}\left\{x^{\frac{3}{2}}-3x+2.3x^{\frac{1}{2}}-2.3\right\}.$$

Ex. 7. $du=lx^2.x^3dx$.

$$\therefore u=lx^2.\frac{x^4}{4}-\int\frac{x^4}{4}.2lx.\frac{dx}{x}$$

$$=\frac{lx^2.x^4}{4}-\frac{1}{2}\left\{lx.\frac{x^4}{4}-\int\frac{x^4}{4}\cdot\frac{dx}{x}\right\}$$

$$=\frac{lx^2.x^4}{4}-\frac{lx.x^4}{8}+\frac{x^4}{32}$$

$$=\frac{x^4}{4}\left\{lx^2-\frac{lx}{2}+\frac{1}{2.4}\right\}.$$

Ex. 8. $du=\dfrac{lx^2dx}{x^3}$.

Assume $\text{P}=lx^2.x^{-2}\ \therefore d\text{P}=-2du+2x^{-2}lx\dfrac{dx}{x}$

$$\therefore u=-\frac{\text{P}}{2}+\int x^{-3}lxdx=-\frac{\text{P}}{2}-\frac{lx.x^{-2}}{2}+\int\frac{x^{-2}}{2}\cdot\frac{dx}{x}$$

$$=-\frac{lx^2.x^{-2}}{2}-\frac{lx.x^{-2}}{2}-\frac{x^{-2}}{2.2}$$

$$=-\frac{x^{-2}}{2}\left\{lx^2+\frac{lx}{1}+\frac{1}{1.2}\right\}.$$

Ex. 9. $du=\frac{dx}{lx^2.x^3}$.

Assume $P=lx^{-1}.x^{-2}$ $\therefore$ $dP=-du-\frac{2dx}{lx.x^3}$,

$\therefore u=-lx^{-1}.x^{-2}-2\int\frac{dx}{lx.x^3}$, which may be integrated by developement.

Ex. 10. $du=vxdx$ where $v=l.\frac{a+x}{a-x}$.

$$v=l(a+x)-l(a-x)\ \therefore\ dv=\frac{dx}{a+x}+\frac{dx}{a-x}=\frac{2adx}{a^2-x^2}$$

$$\therefore u=\frac{vx^2}{2}-\int\frac{x^2}{2}dv=\frac{vx^2}{2}-a\int\frac{x^2dx}{a^2-x^2}$$

$$=\frac{vx^2}{2}-a\int dx\left\{-1+\frac{a^2}{a^2-x^2}\right\}$$

$$=\frac{vx^2}{2}+ax-\frac{a^2v}{2}.$$

Ex. 11. $du=\frac{dx}{x}.lxl^2x$ *.

$$du=l^2x.lxdlx\ \therefore\ u=l^2x.\frac{lx^2}{2}-\int\frac{lx^2}{2}\cdot\frac{dlx}{lx}$$

$$=\frac{l^2x.lx^2}{2}-\frac{lx^2}{4}=lx^2\left\{\frac{l^2x}{2}-\frac{1}{4}\right\}.$$

Ex. 12. $du=\frac{a^xdx}{\sqrt{1+a^{nx}}}$.

Substitute $y=a^x$ $\therefore$ $du=\frac{dy}{la\sqrt{1+y^n}}$, an algebraick form, which may be developed and integrated.

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* It may be necessary to remind the reader that l^2x denotes the logarithm of the logarithm of x, lx^2 the square of the logarithm of x, and $l.x^2$ the logarithm of the square of x.

Ex. 13. $du = dx\int dx\int dx\int dx \cdots$ *ad inf.*

$du = dx \cdot u \therefore dx = \dfrac{du}{u}$ and $x = lu$ and $u = e^x$.

Ex. 14. $du = dx\int dx\int dx \cdots$ to n terms.

Since $\dfrac{x^2}{2} = \int x dx = \int\int dx dx$,

$$\frac{x^3}{1.2.3} = \int\frac{x^2dx}{1.2} = \int\int x dx dx = \int\int\int dx dx dx,$$

$$\frac{x^4}{1.2.3.4} = \int\frac{x^3dx}{1.2.3} = \int\int\frac{x^2dxdx}{1.2} = \int\int\int x dx dx dx = \int\int\int\int dx dx dx dx$$

&c. = &c.

$$\therefore \int dx\int dx\int dx \cdots \text{ to } n \text{ terms} = \frac{x^n}{1.2.3\ldots n}.$$

64. *Circular arc fluents.*

1. $du = \dfrac{dx}{\sin.x}$.

$$\therefore du = \frac{dx.\sin.x}{\sin.^2x} = \frac{-d.\cos.x}{1-\cos.^2x} \therefore u = l\frac{\sqrt{1-\cos.x}}{\sqrt{1+\cos.x}}$$
$$= l.\tan.\tfrac{1}{2}x \text{ (Trig. p. 40).}$$

2. $du = \dfrac{dx}{\cos.x} \therefore$

$$du = \frac{dx\cos.x}{\cos.^2x} = \frac{d\sin.x}{1-\sin.^2x} = l\frac{\sqrt{1+\sin.^2x}}{\sqrt{1-\sin.^2x}} = l\cot.\tfrac{1}{2}\left(\frac{\pi}{2} - x\right).$$

3. $du = \dfrac{dx}{\sin.x\cos.x}$.

$$du = \frac{2dx}{\sin.2x} \therefore \text{(Ex. 1.)},\ u = l\frac{\sqrt{1-\cos.2x}}{\sqrt{1+\cos.2x}} = l.\tan.x.$$

Otherwise. $du = \dfrac{dx}{\cos.^2x} \div \dfrac{\sin.x}{\cos.x} = \dfrac{d.\tan.x}{\tan.x} \therefore u = l.\tan.x.$

4. $du = \dfrac{xdx}{\sin.^2x}$.

Assume $P = \sin.x^{-1} \times x = \cot.x \times x$

$$\therefore d\text{P} = \cot.x dx - \frac{x dx}{\sin.^2 x},$$

$$\therefore u + \text{P} = \int \cot.x dx = \int \frac{\cos.x dx}{\sin.x} = l \sin.x$$

$$\therefore u = l \sin.x - \sin.x^{-1}.x.$$

5. $du = dx \sin.^3 x.$

$du = \sin.^2 x \times - d \cos.x = (\cos.^2 x - 1) d \cos.x.$

$\therefore u = \frac{1}{3}\cos.^3 x - \cos.x.$

6. $du = dx \sin.^4 x.$

$du = -\sin.^3 x d \cos.x =$, by substitution, $-dy.(1-y^2)^{\frac{3}{2}}$

$$\therefore du = -dy.(1-y^2)\sqrt{1-y^2} = -dy\sqrt{1-y^2} + y^2 dy \sqrt{1-y^2}.$$

Assume $\text{P} = y.(1-y^2)^{\frac{3}{2}} \therefore d\text{P} = -du - 3y^2 dy\sqrt{1-y^2},$

$$\therefore \int y^2 dy \sqrt{1-y^2} = -\frac{u+\text{P}}{3} \therefore u = \int -dy\sqrt{1-y^2} - \frac{u+\text{P}}{3},$$

$$\therefore u = -\frac{3}{4}.\int dy \sqrt{1-y^2} - \frac{3}{4} y.(1-y^2)^{\frac{3}{2}}$$

$$= \frac{3}{4}\text{cir. area, arc} = x - \frac{3}{4}.\cos.x \sin.^3 x.$$

7. $du = \cos.^2 x \sin.^3 x dx.$

$du = -\cos.^2 x \sin.^2 x d \cos.x = -\cos.^2 x (1 - \cos.^2 x) d \cos.x,$

$$\therefore u = \frac{1}{5}\cos.^5 x - \frac{1}{3}\cos.^3 x.$$

8. $du = x^2 dx \sin.^{-1} x.$

$$du = \frac{x^3}{3}.\sin.^{-1} x - \int \frac{x^3}{3} d.\sin.^{-1} x$$

$$= \frac{x^3.\sin.^{-1} x}{3} - \frac{1}{3}\int \frac{x^3 dx}{\sqrt{1-x^2}}.$$

Assume $\text{P} = x^2\sqrt{1-x^2} \therefore d\text{P} = 2x dx\sqrt{1-x^2} - \frac{x^3 dx}{\sqrt{1-x^2}}$

$$\therefore \int \frac{x^3 dx}{\sqrt{1-x^2}} = -2\sqrt{1-x^2} - x^2\sqrt{1-x^2}$$

$$\therefore u = \frac{x^3.\sin.^{-1} x}{3} + \frac{1}{3}.(x^2+2)\sqrt{1-x^2}.$$

9. $du = \dfrac{dx}{a + b\cos.x}$ (a greater than b).

Substitute $y=\frac{1}{a+b\cos.x}$ $\therefore a+b\cos.x=\frac{1}{y}$ and $\cos.x$

$$=\frac{1-ay}{by}$$

$$\because \sin.x=\sqrt{1-\left(\frac{1-ay}{by}\right)^2}=\frac{\sqrt{-(a^2-b^2)y^2+2ay-1}}{by}.$$

Hence dx, which $=\frac{-d\cos.x}{\sin.x}$, $=\frac{dy}{y\sqrt{-(a^2-b^2)y^2+2ay-1}}$

$\therefore du=\frac{dy}{\sqrt{-(a^2-b^2)y^2+2ay-1}}$, which is of the same form as Art. 47. Ex. 7.

Assume therefore

$$z=\frac{a}{a^2-b^2}-y \therefore z^2=\frac{a^2}{(a^2-b^2)^2}-\frac{2ay}{a^2-b^2}+y^2$$

$$\therefore -y^2+\frac{2ay}{a^2-b^2}-\frac{1}{a^2-b^2}=\frac{b^2}{(a^2-b^2)^2}-z^2,$$

$$\therefore du=\frac{1}{\sqrt{a^2-b^2}}\cdot\frac{-dz}{\sqrt{\frac{b^2}{a^2-b^2}-z^2}}$$

$$=\frac{1}{\sqrt{a^2-b^2}}\cdot\frac{-\frac{a^2-b^2}{b}dz}{\sqrt{1-\frac{(a^2-b^2)^2z^2}{b^2}}},$$

$$\therefore u=\frac{1}{\sqrt{a^2-b^2}}\cos.^{-1}\frac{(a^2-b^2)z}{b}$$

$$=\frac{1}{\sqrt{a^2-b^2}}\cos.^{-1}\left\{\frac{a}{b}-\frac{(a^2-b^2)y}{b}\right\}$$

$$=\frac{1}{\sqrt{a^2-b^2}}\cos.^{-1}\left\{\frac{a}{b}-\frac{a^2-b^2}{b(a+b\cos.x)}\right\}$$

$$=\frac{1}{\sqrt{a^2-b^2}}\cos.^{-1}\frac{b+a\cos.x}{a+b\cos.x}.$$

MISCELLANEOUS PRAXIS.

1. $\int a^x x dx = \frac{a^x}{la}\left(x - \frac{1}{la}\right).$

2. $\int \frac{a^x dx}{\sqrt{1-a^{2x}}} = \frac{1}{la} \cdot \sin.^{-1} a^x.$

3. $\int \frac{dx}{\sqrt{1+a^x}} = x - \frac{2}{la}\, l\,(\sqrt{1+a^x} + 1).$

4. $\int \frac{dx lx}{x} = \frac{1}{2} lx^2.$

5. $\int \frac{x dx lx}{\sqrt{a^2+x^2}} = \sqrt{a^2+x^2}(lx-1) - al.\frac{\sqrt{a^2+x^2}-a}{x}.$

6. $\int x^3 dx lx^3 = x^4\left\{\frac{lx^3}{4} - \frac{3lx^2}{4^2} + \frac{3.2lx}{4^3} - \frac{3.2.1}{4^4}\right\}.$

7. $\int \frac{dx lx}{(1-x)^2} = \frac{xlx}{1-x} - l \cdot \frac{1}{1-x}.$

8. $\int \frac{dx}{x} l \cdot \frac{1}{1-x} = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \text{\&c.}$

9. $\int v^2 dx$, where $v = l(x + \sqrt{a^2+x^2})$, $= v^2 x - 2v\sqrt{a^2+x^2} + 2x.$

10. $\int dx l.(1+x)^2 = xl.(1+x)^2 + 2l.(1+x) - 2x.$

11. $\int dx . l(1+x)^2 = (1+x)(l(1+x)^2 - 2l(1+x) + 2).$

12. $\int \frac{-dx}{l.\left(\frac{a}{x}\right)^{\frac{1}{2}}}$ may be reduced to $\int \frac{dy}{y^3 ly}.$

13. $\int \frac{dx}{x\sqrt{x}}\, l\,\frac{1}{1-x} = 2x^{-\frac{1}{2}} l(1-x) + 2l.\frac{1+x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}}.$

14. $\int dx \int dx \int \frac{dx}{x} = x^2\left(\frac{lx}{2} - \frac{3}{4}\right).$

15. $\int dx \int dx \int \frac{dx}{2x-x^2} = \frac{x^2}{2} vs^{-1} x + (x-1)l(2-x) - \frac{x}{2}.$

16. $\int dx \sin.^2 x$ = cir. area, coarc = x.

17. $\int dx \cos.^3 x = \sin.x - \frac{1}{3}\sin.^3 x.$

18. $\int dx \operatorname{cosec.} 2x = \frac{1}{2} l \tan.x.$

19. $\int \dfrac{dx}{\sec.x \operatorname{cosec}.x} = \frac{1}{4} vs 2x.$

20. $\int \sin.^4 x \cos.^3 x dx = \dfrac{\sin.^5 x}{5} - \dfrac{\sin.^7 x}{7}.$

21. $\int \tan.^{-1} x dx = \tan.^{-1} x . x - l.(1+x^2)^{\frac{1}{2}}.$

22. $\int \dfrac{xdx}{\sqrt{1-x^2}} \sin.^{-1} x = -\sqrt{1-x^2} \sin.^{-1} x + x.$

23. $\int \dfrac{dx}{(1-x^2)^{\frac{3}{2}}} \sin.^{-1} x = \dfrac{x.\sin.^{-1} x}{\sqrt{1-x^2}} + l.\sqrt{1-x^2}.$

24. $\int \tan.^{-1} x^2 . xdx = \frac{1}{2} \tan.^{-1} x^2 .(x^2+1) - \tan.^{-1} x . x$
$+ \frac{1}{2} l.(1+x^2).$

25. $\int \dfrac{dx}{1-\tan.^2 x} = \frac{1}{2} x + \frac{1}{4} l. \dfrac{1+\tan.x}{1-\tan.x}.$

26. $\int \dfrac{lxdx}{(1-x)^2} = \dfrac{xlx}{1-x} + l(1-x).$

27. $\int \dfrac{lxdx}{1-x} = -lxl(1-x) + \int l(1-x) \dfrac{dx}{x}.$

28. $\int \dfrac{l(1-x)dx}{x^{\frac{3}{2}}} = -\dfrac{2l(1-x)}{x^{\frac{1}{2}}} - 2l. \dfrac{1+x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}}.$

29. $\int \dfrac{lx.dx}{\sqrt{1-x^2}}$ } may be developed.
30. $\int \dfrac{lx^2.dx}{\sqrt{1-x^2}}$ }

31. $\int \dfrac{(1-x^3)^{\frac{1}{3}} dx}{x^5} = -\dfrac{(1-x^3)^{\frac{4}{3}}}{4x^4}.$

32. $\int \dfrac{x^{\frac{3}{2}} dx \sqrt{2a-x}}{(a-x)^2} = \dfrac{(2a-x)\sqrt{2ax-x^2}}{a-x} + a \sin.^{-1} \dfrac{a-x}{a}$
$+ al. \dfrac{a-\sqrt{2ax-x^2}}{a-x}.$

33. $\int \dfrac{x^2+1}{x^2-1} . \dfrac{dx}{\sqrt{1-ax^2+x^4}}$, when a is greater than 2,

$$= \frac{1}{\sqrt{a-2}} \sec.^{-1} \frac{x^2-1}{x\sqrt{a-2}}.$$

34. $$\int \frac{1+x^{\frac{1}{2}}-x^{\frac{2}{3}}}{1+x^{\frac{1}{3}}} dx = -\frac{3x^{\frac{4}{3}}}{4} + \frac{6x^{\frac{7}{6}}}{7} + x - \frac{6x^{\frac{5}{6}}}{5} + 2x^{\frac{1}{2}} - 6x^{\frac{1}{6}}$$

$$+ 6 \tan.^{-1} x^{\frac{1}{6}}.$$

35. $$\int \frac{dx}{x(1+x)^2(1+x+x^2)} =$$

$$l. \frac{x\sqrt{1+x+x^2}}{(1+x)^2} + (1+x)^{-1} - \frac{1}{\sqrt{3}} \tan.^{-1} \frac{2x+1}{\sqrt{3}}.$$

36. $$\int \frac{dx}{x^8+x^7-x^4-x^3} = \frac{2-2x-5x^2}{4x^2(1+x)} + \tfrac{1}{8} l. \frac{x^2-1}{x^2+1}$$

$$+ l. \frac{x+1}{x} - \tfrac{1}{4} \tan.^{-1} x.$$

CHAPTER III.

On finding the first Fluxions of any Combination of Functions.

1. THEOR. *If* u = fx, *the general form of a series which is the developement of* U = f(x + h), *where* h *is independent of* x, *must be* $U = u + ph + qh^2 + rh^3 +$ *&c.*, *where the first term is* fx *or* u, *the succeeding terms contain the ascending positive and integral powers of* h, *and the coefficients* p, q, r, *&c. are functions of* x *independent of* h.

The axiom upon which the demonstration rests is this; that $f(x+h)$ when expanded possesses the same *general* properties as before expansion.

Hence the first term must be fx, because one property of $f(x+h)$ is, that when $h=0$, it becomes fx.

From the same principle it follows that the sum of the remaining terms is a multiple of h; where h cannot be fractional, for then $f(x+h)$ would be infinite when $h=0$.

Nor can h be a radical quantity; for, if possible, let there be a term of the form $B \times h^{\frac{m}{n}}$ or $B\sqrt[n]{h^m}$.

It is evident, that so long as x is indeterminate, there will be the same radicals and under the same form in $f(x+h)$ as in fx (See Note 1); hence fx and $f(x+h)$ must have the same number of values, viz. as many as there are units in the denominator of the radical exponent (Note 2). If then

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Note 1. If $fx=x^{\frac{1}{2}}$, $f(x+h)=f(x+h)^{\frac{1}{2}}$; if $fx=x^{\frac{1}{3}}$, $f(x+h)=(x+h)^{\frac{1}{3}}$; if $fx=bx^2+\frac{a}{\sqrt{x}}$, $f(x+h)=b.(x+h)^2+\frac{a}{\sqrt{x+h}}$, &c. &c.

Note 2. If $fx=x^{\frac{1}{2}}$, each has two, for $\sqrt{x}=\pm x^{\frac{1}{2}}$; if $fx=x^{\frac{1}{3}}$ each has three, viz. the cube root of $x \times$ the three cube roots of unity. Alg. Part ii. Art. 266.

we suppose that there is a term of the form $\text{B} \times \sqrt[n]{h^m}$, $f(x+h)$ is irrational, and has n different values; hence it follows that fx must also have n different values, and by their combination the series $fx + ph + qh^2 + \ldots + \text{B}h^{\frac{m}{n}} +$ &c. will have more than n values, or the developed function possesses more values than when undeveloped, which is contrary to the axiom (Note 3).

Assume then $\text{U} = u + \text{P}h$, P being a function both of x and h; then for the same reason that we may assume $\text{U} = u + \text{P}h$, we may likewise assume $\text{P} = p + \text{Q}h$, provided that p is the value of P when $h = 0$, and consequently a sole function of x, and Q a function both of x and h. Similarly it may be assumed that $\text{Q} = q + \text{R}h$, where $q = \text{Q}$ when $h = 0$ and $\text{R} = \varphi(x, h)$; &c. Hence we have

$$\left.\begin{aligned} \text{U} &= u + \text{P}h \\ \text{P} &= \text{P} + \text{Q}h \\ \text{Q} &= q + \text{R}h \\ \text{R} &= r + sh \\ \text{\&c.} &= \text{\&c.} \end{aligned}\right\} \begin{aligned} \text{therefore } \text{U} &= u + h.\{p + \text{Q}h\} = u + ph + \text{Q}h^2 \\ &= u + ph + h^2.\{q + \text{R}h\} \\ &= u + ph + qh^2 + h^3\{r + sh\} \\ &= u + ph + qh^2 + rh^3 + sh^4 + \text{\&c.} \end{aligned}$$

In this Theorem the variable x is supposed to be *indeterminate:* if we assign a particular value to it, it may cause a radical to disappear in the function which remains in the developement, and consequently the reasoning of this article ceases to be applicable. Thus let $fx = cx^2 + (x + b)\sqrt{x - a}$, then if we suppose $x = a$, $f(x+h)$ contains a radical, viz. $(a + b)\sqrt{h}$, which disappears in fx.

Cor. It follows, from the doctrine of limits, that if $u = \text{F}x$, $\text{F}(x + h) = \text{F}x + \frac{du}{dx} h + q\ h^2 + r\ h^3 +$ &c. where q, r, &c. are functions of x independent of h.

Note 3. Let the series, if possible, contain $\text{B} \times \sqrt[3]{h}$, and let the cube roots of unity be P, Q, R, then

$$\left.\begin{aligned} f(x+h) &= fx + ph + qh^2 + \ldots + \text{B}h^{\frac{1}{3}} \times \text{P} + \text{\&c.} \\ &= fx + ph + qh^2 + \ldots + \text{B}h^{\frac{1}{3}} \times \text{Q} + \text{\&c.} \\ &= fx + ph + qh^2 + \ldots + \text{B}h^{\frac{1}{3}} \times \text{R} + \text{\&c.} \end{aligned}\right\}$$

But since $f(x+h)$ contains a cube root, fx also contains a cube root, and has three values, which, substituted in these equations, will produce nine different series to express a function which has only three values.

For $\frac{U-u}{h} = p+q\,h + \&c.$, therefore, taking the limiting ratio, $\frac{du}{dx}$ may be substituted for p.

2. *The quantity* h *may be always assumed so small that any term of the developement may be greater than the sum of its succeeding terms; and the same will continue to be true, if* h *be taken any magnitude whatever less than the assumed magnitude.*

Let it be required to prove this of the term qh^2.

"Since the sum of the remaining terms Rh^3 is a function of h, we may consider it as an ordinate of a curve corresponding to the abscissa h; and since it becomes $=0$ when $h=0$, it follows that this curve must cut the axis at the origin of the abscissa; and unless there is a singular point at the origin *, which can only take place for particular values of x, the curve must be continuous up to that point, and consequently must gradually approach the axis, and will approach it by a distance less than any given quantity, so that we can always assume an abscissa h such that its ordinate Rh^3, without being $=0$, shall be less than any given quantity, and if we take any other abscissa less than h, since the curve approaches the axis, its ordinate must be less than the ordinate of h, and consequently less than any given quantity. The quantity h then may be assumed so small that qh^2 may be greater than Rh^3, and the same will continue to be true if h be in any degree diminished."—*Theorie Fonct. Anal.* p. 14.

The truth of this proposition, which will be frequently referred to in the following chapters, may be established at once by the doctrine of limits, if the function be expanded in a series composed of ascending powers of h.

For $U = u+ph+h^2\left\{q+Rh\right\}$, in which, if h be indefinitely diminished, Rh in the limit becomes less than q, or q greater than Rh; and consequently qh^2 becomes greater than Rh^3 the sum of the remaining terms.

3. *If there be three series of the form*

$$\left.\begin{array}{l} A_1+B_1\,h+C_1\,h^2+\&c. \\ A_2+B_2\,h+C_2\,h^2+\&c. \\ A_3+B_3\,h+C_3\,h^2+\&c. \end{array}\right\}$$

such that the value of the second always lies between the values of the first and third, and if A_3 *be made equal to* A_1, A_2 *shall also be equal to* A_1 *or* A_3.

* This will be explained in the 12th Chapter.

For take h so small that the second term may be greater than the sum of the remaining terms; and suppose the series to become $\left.\begin{array}{l} A_1 + s' \\ A_2 + s'' \\ A_3 + s''' \end{array}\right\}$.

If when A_3 becomes equal to A_1, A_2 is not equal to A_3 or A_1, let $A_2 = A_3 + d$ or $= A_1 + d$; and subtracting the third from the second, and also the second from the first, we have $\left\{\begin{array}{l} \phantom{\text{and}} + d + s'' - s''' \\ \text{and} - d + s' - s'' \end{array}\right\}$; which (ex hyp.) are necessarily both positive; but h may be assumed so small that $s'' - s'''$ and $s' - s''$ may be less than d, so long as d is a finite magnitude; and making this assumption, we have $- d + s''' - s'$ negative and $+d + s'' - s'''$ positive, which is contrary to the supposition; hence d cannot be finite, and $A_2 = A_1$ or A_3.

This also follows immediately from the principle of limits; for the last ratio of the three series, when h is indefinitely diminished, is $A_1 : A_2 : A_3$; and if A_1 is made equal to A_3, it is manifest that A_2, which is between them, must be equal to either of them.

4. *Required to investigate the law by which the coefficients of the expanded binomial are derived from the coefficient of its second term.*

Let fx become $f(x+h+k)$, where h and k are independent quantities, then the developement of this is the same whether we suppose it to arise from x in fx becoming $x+(h+k)$ or from x in $f(x+h)$ becoming $x+k$.

On the first supposition $f(x+(h+k)) = fx + p.(h+k) + q.(h+k)^2 + r.(h+k)^3 + \&c.$ and to find the developement on the second hypothesis, we have $f(x+h) = fx + ph + qh^2 + rh^3 + \&c.$, in which, if we suppose x to become $x+k$, each term of the series will admit of a developement,

$$\left.\begin{array}{lll} & fx \text{ becomes} & fx + p\,k + q\,k^2 + r\,k^3 + \&c. \\ \text{let} & p \text{ become} & p + p'k + p''k^2 + p'''k^3 + \&c. \\ & q \quad — & q + q'k + q''k^2 + q'''k^3 + \&c. \\ & r \quad — & r + r'k + r''k^2 + r'''k^3 + \&c. \\ & \&c. \; — & + \&c. \end{array}\right\} \text{hence the last equation becomes}$$

$$\left.\begin{array}{rl} f(x+h+k) = fx + p\,k + q\,k^2 + r\,k^3 + \&c. & \\ +h\left\{p + p'\,k + p''\,k^2 + \&c.\right\} & \\ +h^2\left\{q + q'\,k + \&c.\right\} & \\ +h^3.\left\{r + \&c.\right\} & \\ + \&c. & \end{array}\right\}$$

which is to equal $fx+p.(h+k)+$ &c. and equating terms which include like powers of h and k, (Alg. 346), we have $qk^2+p'hk+qh^2=q.(h+k)^2$; $rk^3+p''k^2h+q'kh^2+rh^3=r(h+k)^3$; &c. hence $p'=2q$; $q'=3r$; $r'=4s$; &c.

Lagrange calls p the *first* derived function of fx, and denotes it by $f'x$; hence p', q', r', &c. are the first derived functions of p, q, r, &c.; and thus p' may be considered with reference to fx as its *second* derived function; it is denoted by $f''x$, i. e. if $p=f'x$, in this notation, p' shall $=f''x$.

Similarly denote the first derived function of $f''x$, which may be called the *third* derived function of fx by $f'''x$; then, since $q=\frac{p'}{2}=\frac{f''x}{2}$, it follows that $q'=\frac{f'''x}{2}$; and therefore $r=\frac{f'''x}{2.3}$. In the same manner it may be shown that $s=\frac{f^{\text{iv}}x}{1.2.3.4}$, &c. &c.; hence we have

$$f(x+h)=fx+f'x\cdot\frac{h}{1}+f''x.\frac{h^2}{1.2}+f'''x.\frac{h^3}{1.2.3}+\&c.$$

Ex. Let $fx=\frac{1}{x}$, then $f(x+h)=\frac{1}{x+h}=\text{U}$, therefore

$\text{P},=\frac{\text{U}-u}{h},=\frac{1}{h}\left\{\frac{1}{x+h}-\frac{1}{x}\right\}=-\frac{1}{x.(x+h)}$, therefore p, $=(\text{P}),=-\frac{1}{x^2}$; therefore $\text{Q}=\frac{1}{h}\left\{\frac{\text{U}-u}{h}-p\right\}\ldots\ldots=\frac{1}{h}\left\{\frac{1}{x^2}-\frac{1}{x.(x+h)}\right\}=\frac{1}{x^2.(x+h)}$, therefore $q=\frac{1}{x^3}$.

Similarly $r=-\frac{1}{x^4}$, &c. &c., and we have

$$\left.\begin{aligned}\frac{1}{x+h}&=\frac{1}{x}-\frac{1}{x+h}\\&=\frac{1}{x}-\frac{h}{x^2}+\frac{h^2}{x.(x+h)}\\&=\frac{1}{x}-\frac{h}{x^2}+\frac{h^2}{x^3}-\frac{h^3}{x^3.(x+h)}\\&=\&c.\end{aligned}\right\}\text{which is the same result as that obtained by actual division.}$$

5. *Lagrange's notation explained.*

When $u = F(x, y)$ or is a function of two independent variables, we may suppose the changes to take place separately; and Lagrange denotes, as before, the 1st, 2d, 3d, &c. derived functions which arise from the change of x by $F'(x,y)$, $F''(x,y)$, $F'''(x,y)$, &c. The 1st, 2d, 3d, &c. derived functions arising from the change of y, he denotes by $F_{\prime}(x,y)$, $F_{\prime\prime}(x,y)$, $F_{\prime\prime\prime}(x,y)$, &c. i. e. if y in $F(x,y)$ becomes $y+k$, we shall have $F(x,y+k) = F(x, y) + F_{\prime}(x,y)\dfrac{k}{1} + F_{\prime\prime}(x, y)\dfrac{k^2}{1.2}$ + &c.

$F'_{\prime}(x, y)$ denotes a function which has been derived twice, the first from $F(x, y)$, on the supposition that x alone varies, and the second derived from $F'(x,y)$ on the supposition that y alone varies.

$F''_{\prime}(x, y)$ is a function which has been derived twice successively from $F(x,y)$ on the supposition that x alone varies, and then derived a third time from $F''(x,y)$ on the supposition that y alone varies, &c. &c.

We shall occasionally make use of this notation.

6. *Required to investigate the series which is the developement of a function of two independent variables.*

Let $u = F(x,y)$, and let x and y become $x+h$, and $y+k$ where h and k are independent quantities.

The first term of the series is manifestly $F(x,y)$ or u, and the remaining terms are certain derived functions of x and y multiplied into h, k, h^2, hk, k^2, h^3, &c. &c.

Then, since x and y are independent quantities, we may suppose their changes to take place separately.

1st. Let x become $x+h$, then (Art. 4.)

$$F(x+h, y) = u + u'\frac{h}{1} + u''\frac{h^2}{1.2} + u'''\frac{h^3}{1.2.3} + \&c.$$

Next, in this equation, suppose y to become $y+k$, then each term will receive a change,

$F(x+h, y)$ becomes $F(x+h, y+k)$

$$u \text{ becomes } u + u_{\prime}\frac{k}{1} + u_{\prime\prime}\frac{k^2}{1.2} + u_{\prime\prime\prime}\frac{k^3}{1.2.3} + \&c.$$

$$u' \quad - \quad u' + u'_{\prime}\frac{k}{1} + u'_{\prime\prime}\frac{k^2}{1.2} + \&c.$$

u'' becomes $\qquad u'' + u_{,}'' \dfrac{k}{1} + \&c.$

$u''' \quad — \qquad u'''' + \&c.$

&c. — &c.

Hence $\text{F}(x+h,\ y+k)$

$$= u + u_{,}\frac{k}{1} + u_{,,}\frac{k^2}{1.2} + u_{,,,}\frac{k^3}{1.2.3} + \&c.$$

$$+\frac{h}{1}\left\{u' + u_{,}'\frac{k}{1} + u_{,,}'\frac{k^2}{1.2} + \&c.\right\}$$

$$+\frac{h^2}{1.2}\left\{u'' + u_{,}''\frac{k}{1} + \&c.\right\}$$

$$+\frac{h^3}{1.2.3}\left\{u''' + \&c.\right\}$$

$$+ \&c.$$

$$= u + (u'h + u_{,}k) + 1.2(u''h^2 + u_{,}'\,2hk + u_{,,}k^2)$$
$$+ 1.2.3(u'''h^3 + u_{,}''\,3h^2k + u_{,,}'\,3hk^2 + u'''\,k^3 + \&c.$$

The *terminus generalis* is $\dfrac{h^m k^n}{1.2\ldots m \times 1.2\ldots n} + \text{F}_n^m\,(x,\ y)$.

This investigation may be extended to a function of three or more variables.

Cor. Since it may be shown by the doctrine of limits that $u' = \dfrac{du}{dx}$ and $u_{,} = \dfrac{du}{dy}$, we have $\text{F}(x+h, y+k)$ . . . .

$$= u + \frac{du}{dx}h + \frac{du}{dy}k + 1.2(u''h^2 + u_{,}'2hk + u_{,,}k^2) + \&c.$$

Similarly it may be shown that if $u = \text{F}(x,\ y,\ z)$ . . . . . .

$$\text{F}(x+h,\ y+k,\ z+l) = u + \frac{du}{dx}h + \frac{du}{dy}k + \frac{du}{dz}l + \&c.$$

7. *Lagrange's Theory of Functions.*

The preceding articles contain the principles of Lagrange's Theory of Functions. It was his object to establish a system of calculation independent of the doctrine both of limits and of infinitely small quantities. Having demonstrated the general theorem, that any simple function as $\text{F}(x+h)$, whatever be its nature, algebraick or transcendental, is developeable in a series of the form $\text{F}x + ph + qh^2 + \&c.$, he then

shows that the coefficients q, r, &c. may be all successively obtained by a certain law from p, the coefficient of the second term; so that if we can find p, all the other derived functions may be deduced from it.

It has already appeared (Art. 1. Cor.) that the quantity p is the same function of the variable as that which we represent by $\frac{du}{dx}$, and in the following chapter all the coefficients will be represented in the fluxional notation. If then p can be obtained from its function in all cases by an algebraick process, the necessity of introducing into the science the doctrine of limits is superseded, and the science itself is thus brought back to what are termed the ordinary operations of algebra, multiplication, division, involution and evolution.

Now all simple functions are included in one of the following forms, x^n, a^x, $\text{L}x$, $\sin.x$, $\cos.x$, &c., omitting the inverse functions $\text{L}^{-1}x$, $\sin.^{-1}x$, &c. as being easily deducible from $\text{L}x$, $\sin.x$, &c.

Of these forms the three first present no difficulty; for $(x+h)^n=x^n+nx^{n-1}.h+$ &c.; $a^{x+h}=a^x+la.a^x.h+$ &c; and $\text{L}(x+h)=\text{L}x+\frac{1}{la.x}.h-\frac{1}{la.x^2}\frac{h^2}{1.2}+$ &c.; and consequently their first derived functions are nx^{n-1}, $la.a^x$, and $\frac{1}{la.x}$; but with respect to *circular* functions, though the fluxional calculus enables us to show that

$$\left.\begin{aligned}\sin.(x+h)&=\sin.x+\cos.x\frac{h}{1}-\sin.x\frac{h^2}{1.2}-\text{&c.}\\ \cos.(x+h)&=\cos.x-\sin.x\frac{h}{1}-\cos.x\frac{h^2}{1.2}+\text{&c.}\end{aligned}\right\}$$

(Ch. 4. Taylor's Theor. Ex. 2.), yet is there no *direct* method of expanding these functions which is strictly algebraical. Lagrange assumes that $\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}}$ and $\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2}$ are the proper analytical symbols of $\sin.x$ and $\cos.x$ (vid. Ch. 2. Art. 36.), and expanding these as exponentials, he finds the first derived functions of $\sin.x$ and of $\cos.x$ to be $\cos.x$ and $-\sin.x$ respectively.

Other writers, not understanding how the introduction of imaginary symbols can contribute to the rigour of the demonstration, have deduced the same results from the well known series for the sine and the cosine in terms of the arc; thus

$$\left.\begin{array}{l}\sin.x = x - \dfrac{x^3}{1.2.3} + \dfrac{x^5}{1.2.3.4.5} - \&c.\\ \cos.x = 1 - \dfrac{x^2}{1.2} + \dfrac{x^4}{1.2.3.4} - \&c.\end{array}\right\} \text{(4. 4. Ex. 4.); hence,}$$

$$\begin{aligned}\sin.(x+h), \text{ which } &= \sin.x \cos.h + \cos.x \sin.h,\\ &= \sin.x \left\{1 - \frac{h^2}{1.2} + \frac{h^4}{1.2.3.4} - \&c.\right\}\\ &\quad + \cos.x \left\{h - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} - \&c.\right\}\\ &= \sin.x + \cos.x.h - \&c.\end{aligned}$$

and $\cos.(x+h)$ may be developed by the same means. But the process of expanding algebraically the sine and the cosine in terms of the arc is far from being satisfactory. Vid. Theor. Fonct. Anal. p. 24. La Croix, Calc. Diff. and Integ. tom. i. p. 159.

The reader will find some important objections to Lagrange's Theory of Functions in *Woodhouse's Principles of Analytical Calculation*, preface, pp. 18—25.

It is with respect to circular functions that the doctrine of limits possesses such peculiar advantages.

We first show (1.31) that $\dfrac{\text{U}-u}{h} = \dfrac{\cos.x \sin.h - vs.h \sin.x}{h}$ by a simple algebraick process; and if the use of infinitesimals be considered objectionable, we must divide the expression for $\dfrac{\text{U}-u}{h}$ into two, and find the ultimate value of each.

Now the ultimate value of $\dfrac{\sin.h}{h}$ may be shown $=1$, either as in (1.31), or from the axiom that an arc is greater than its sine and less than its tangent, and consequently $\sin.h$, h, $\tan.h$ or $\dfrac{\sin.h}{\cos.h}$ are in the order of their magnitudes; but ultimately $\cos.h=1$, therefore in the limit h is between $\sin.h$ and $\sin.h$, and the limit of $\dfrac{\sin.h}{h}=1$ and is finite.

And to find the ultimate value of $\frac{vs.h}{h}$, we have, since chord h is less than h, $\frac{vs.h}{h}$ less than $\frac{vs.h}{\text{chord } h}$, or, from similar triangles, less than $\frac{\text{chord } h}{\text{dia.}}$, but ultimately $\frac{\text{chord } h}{\text{dia.}}=0$, therefore $\frac{vs.h}{h}=0$; and the limit of $\frac{\text{U}-u}{h}$ or $\frac{du}{dx}=\cos.x$.

8. *Of three quantities the first is an explicit function of the second, and the second an explicit function of the third; required to find the fluxion of the first in terms of the fluxion of the third.*

Let $u=\text{F}y$ and $y=fx$, then shall $du=\frac{du}{dy}\frac{dy}{dx}dx$.

For let x become $x+h$, y — $y+k$, u — U, then $\text{U}-u=\text{F}(y+k)-\text{F}y$

$$=\frac{du}{dy}k+qk^2+rk^3+\&c.$$

(Theor. 1. Cor.), and dividing by h, $\frac{\text{U}-u}{h}=\frac{du}{dy}\frac{k}{h}+\frac{qk^2}{h}+\&c.$, and taking these ratios in their limit, for u and x are necessarily dependent quantities, we have

$$\frac{du}{dx}=\frac{du}{dy}\frac{dy}{dx} \text{ or } du=\frac{du}{dy}\frac{dy}{dx}dx.$$

Otherwise. Since y is a sole function of x, $\frac{dy}{dx}dx=dy$, therefore $\frac{du}{dy}\frac{dy}{dx}dx=\frac{du}{dy}dy$, which $=du$ because u is a sole function of y.

Ex. 1. $u=y^m$ and $y=a+bx+cx^2+\&c.$

$\therefore \frac{du}{dy}=my^{m-1}$, and $\frac{dy}{dx}=b+2cx+\&c.$

$\therefore du=my^{m-1}\{b+2cx+\&c.\}dx$

$\quad =m.(a+bx+cx^2+\&c.)^{m-1}\{b+2cx+\&c.\}dx.$

Ex. 2. $u=y\sqrt{a^2-y^2}$, and $y=\sqrt{a^2-x^2}$;

$\therefore \frac{du}{dy}=\sqrt{a^2-y^2}-\frac{y^2}{\sqrt{a^2-y^2}}=\frac{a^2-2y^2}{\sqrt{a^2-y^2}}=\frac{2x^2-a^2}{x}$; and

$\frac{dy}{dx}=\frac{-x}{\sqrt{a^2-x^2}} \therefore du=\frac{2x^2-a^2}{x}\times\frac{-x}{\sqrt{a^2-x^2}}\times dx \ldots\ldots$

$=\frac{(a^2-2x^2)\,dx}{\sqrt{a^2-x^2}}$, which is the same result as if we had first substituted $\sqrt{a^2-x^2}$ for y in the equation $u=y\sqrt{a^2-y^2}$, and then differentiated.

9. *Of three quantities the first is an implicit function of the second and third, and the second is an explicit function of the third; required the fluxion of the first in terms of the fluxion of the third.*

Let $u=F(x,y)$, and $y=fx$; then shall

$$du=\frac{du}{dx}dx+\frac{du}{dy}\frac{dy}{dx}dx.$$

For (Art. 6. Cor.) $U=u+\left(\frac{du}{dx}h+\frac{du}{dy}k\right)+\&c.$

therefore $\frac{U-u}{h}=\frac{du}{dx}+\frac{du}{dy}\frac{k}{h}+\&c.$;

and diminishing h and k without limit, we have

$$\frac{du}{dx}=\frac{du}{dx}+\frac{du}{dy}\frac{dy}{dx}, \text{ or } du=\frac{du}{dx}dx+\frac{du}{dy}\frac{dy}{dx}dx.$$

The du in the first member of this equation represents the *total* fluxion of u, because it arises from taking the whole increment of the function.

The advantage which we derive from this and the preceding expression for du is, that they enable us to find du in terms of dx without first eliminating y. They are also of considerable use in differentiating complicated functions, in which it frequently facilitates the calculation to substitute simple quantities for different parts of the function, as in Ch. 1. Art. 26.

The connexion between the two theorems will be best seen by applying them to the same example. Take Ex. 2. which we must now have under the form . . . .

Ex. $u = \sqrt{a^2 - x^2}.\sqrt{a^2 - y^2}$, and $y = \sqrt{a^2 - x^2}$,
or $u = yx$, and $y = \sqrt{a^2 - x^2}$;

$\therefore \frac{du}{dx}dx = ydx$. Also $\frac{du}{dy} \times \frac{dy}{dx}dx = x \times \frac{-xdx}{\sqrt{a^2-x^2}}$ $\therefore$ by the theorem $du = ydx - \frac{x^2dx}{\sqrt{a^2-x^2}} = \sqrt{a^2-x^2}dx - \frac{x^2dx}{\sqrt{a^2-x^2}}$

$= \frac{(a^2-2x^2)dx}{\sqrt{a^2-x^2}}$.

Ex. 2. Let $x^2+y^2=a^{\frac{y}{x}}$; and let it be required to compare the fluxion of a function, as $x^3 + axy + y^3$ with the fluxion of x.

In this example we do not know the form of the function $y = fx$, nor can we find it; but the truth of the theorem is not affected by this, and we have $u = x^3 + axy + y^3$, and consequently $\frac{du}{dx}dx = (3x^2 + ay)dx$ and $\frac{du}{dy} = ax + 3y^2$.

Also to find $\frac{dy}{dx}dx$, we have $2xdx + 2ydy$

$= la.a^{\frac{y}{x}}\left\{\frac{dy}{x} - \frac{ydx}{x^2}\right\}$, or $\left\{2x + \frac{la.a^{\frac{y}{x}}y}{x^2}\right\}dx$

$= \left\{\frac{la.a^{\frac{y}{x}}}{x} - 2y\right\}dy$, and therefore $\frac{dy}{dx} = \frac{2x^3 + la.a^{\frac{y}{x}}y}{la.a^{\frac{y}{x}}x - 2x^2y}$;

and $du = (3x^2+ay)dx + \frac{2x^3 + la.a^{\frac{y}{x}}y}{la.a^{\frac{y}{x}}x - 2x^2y} \cdot (ax+3y^2)dx$.

10. *If* $u = \mathrm{F}(p, r, s, \&c.)$ *where* p, r, s, &c. *are functions of the same variable* x, $du = \frac{du}{dp}\frac{dp}{dx}dx + \frac{du}{dr}\frac{dr}{dx}dx + \frac{du}{ds}\frac{ds}{dx}dx$ + &c.

For let π, ϱ, σ, &c. be the increments of p, r, s, &c. corresponding to h the increment of x; then (6. Cor.).

$\mathrm{U} = u + \left(\frac{du}{dp}\pi + \frac{du}{dr}\varrho + \frac{du}{ds}\sigma + \&c.\right)$ + &c. the remaining terms containing functions of π, ϱ, σ, &c. which are at least of two dimensions;

therefore $\frac{\text{U}-u}{h}=\frac{du}{dp}\,\frac{\pi}{h}+\frac{du}{dr}\,\frac{\varrho}{h}+\frac{du}{s}\,\frac{\sigma}{h}+$ &c. + &c.; and taking these ratios in the limit,

$$\frac{du}{dx}=\frac{du}{dp}\,\frac{dp}{dx}+\frac{du}{dr}\,\frac{dr}{dx}+\frac{du}{ds}\,\frac{ds}{dx}+\text{\&c.}$$

$$\text{or } du=\frac{du}{dp}\,\frac{dp}{dx}dx+\frac{du}{dr}\,\frac{dr}{dx}dx+\frac{du}{ds}\,\frac{ds}{dx}dx+\text{\&c.}$$

Or the total fluxion of u is equal to the sum of its fluxions taken partially with respect to p, r, s, &c.

Cor. 1. If $p=x$, $du=\frac{du}{dx}dx+\frac{du}{dr}\,\frac{dr}{dx}dx$, as in art. 9.

Cor. 2. If u instead of being a function of x and r is expressed in terms of r, solely; *i. e.* if $u=\text{F}r$ and $r=fx$, $\frac{du}{dx}dx=0$, and $du=\frac{du}{dr}\,\frac{dr}{dx}dx$, as in art. 8.

11. *The fluxions of two equal functions of different variables are equal.*

This follows immediately from the definition, Ch. I. Art. 7; and it may also be proved thus;

Let $\left.\begin{array}{l} u=\text{F}x,\ \text{U}=\text{F}(x+h) \\ v=fy,\ \text{V}=f(y+k) \end{array}\right\}$ where h and k must be so assumed, that

$$\text{F}(x+h)=f(y+k).$$

By Theor. Art. 1. Cor., $\text{U}=u+\frac{du}{dx}h+qh^2+$ &c.

$$\text{V}=v+\frac{du}{dy}k+q'k^2+\text{\&c.}$$

therefore, diminishing the increments indefinitely,

$\frac{du}{dx}h=\frac{dv}{dy}k$ and $\frac{du}{dx}=\frac{dv}{dy}\cdot\frac{k}{h}=\frac{dv}{dy}\cdot\frac{dy}{dx}$, therefore

$du=\frac{dv}{dy}\,\frac{dy}{dx}dx=$ (3. 8.) dv.

In this last step we assume that y either is or may be considered to be a function of x.

12. *The fluxion of the sum of any number of functions of different dependent variables is equal to the sum of their fluxions.*

Let $u=t+v+w+$ &c., where $t=\text{F}x$, $v=fy$, $w=\varphi z$, &c.;

suppose u to become U in consequence of x becoming $x+h$,

$$\begin{array}{ll} y \cdots\cdots & y+k, \\ z \cdots\cdots & z+l, \\ \&c. \cdots\cdots & \&c.; \end{array}$$

then $\text{U}=\text{F}(x+h)+f(y+k)+\varphi(z+l)+$ &c.

$$\left.\begin{array}{l} =\text{F}x+fy+\varphi z+\ \&c.\ (u) \\ +\dfrac{dt}{dx}h+qh^2+\&c. \\ +\dfrac{dv}{dy}k+q'k+\&c. \\ +\dfrac{dw}{dz}l+q''l^2+\&c. \\ +\&c. \end{array}\right\}$$

Where the coefficients of the increments are independent of the increments.

Subtract u, divide by h and take the ratios in their limit;

then $\dfrac{du}{dx}=\dfrac{dt}{dx}+\dfrac{dv}{dy}\dfrac{dy}{dx}+\dfrac{dw}{dz}\dfrac{dz}{dx}+$ &c.;

therefore $du=\dfrac{dt}{dx}dx+\dfrac{dv}{dy}\dfrac{dy}{dx}dx+\dfrac{dw}{dz}\dfrac{dz}{dx}dx+$ &c.

$=dt+dv+dw+$ &c.

13. *Required to differentiate the product of two functions of the same variable.*

$$\left.\begin{array}{l}\text{Let } u=\text{F}x,\ \text{U}=\text{F}(x+h) \\ \quad v=fx,\ \text{V}=f(x+h)\end{array}\right\}\text{then}\left\{\begin{array}{l}\text{U}=u+ph+qh^2+\&c. \\ \text{V}=v+p'h+q'h^2+\&c.\end{array}\right.$$

$$\begin{array}{rl}\text{therefore UV}= & uv+pvh+qvh^2+\&c. \\ & +p'uh+pp'h^2+\&c. \\ & +q'uh^2+\&c. \\ & +\&c.;\end{array}$$

therefore $\dfrac{\text{UV}-uv}{h}=pv+p'u+\left.\begin{array}{l}qv \\ +pp' \\ +q'u\end{array}\right\}h+$ &c.;

and diminishing h indefinitely, we have

$\dfrac{d.uv}{dx}=pv+p'u=\dfrac{du}{dx}v+\dfrac{dv}{dx}u$, or $d.uv=vdu+udv$, which contains the same precept as Ch. 1. Art. 16.

14. *Required to differentiate the quotient of two functions of the same variable.*

Here incre. of $\frac{u}{v} = \frac{\text{U}}{\text{V}} - \frac{u}{v} = \frac{u + ph + qh^2 + \&c.}{v + p'h + q'h^2 + \&c.} - \frac{u}{v} =$ (neglecting the higher powers of h) $\frac{(pv - p'u)h}{v.(v + p'h)}$, therefore

$\frac{\text{inc.} \frac{u}{v}}{h} = \frac{p - p'\frac{u}{v}}{v + p'h}$; and diminishing h indefinitely, $\frac{d.\frac{u}{v}}{dx}$

$= \frac{p - p'\frac{u}{v}}{v} = \frac{\frac{du}{dx} - \frac{dv}{dx}\frac{u}{v}}{v}$, therefore $d.\frac{u}{v} = \frac{du}{v} - \frac{udv}{v^2}$ as in Ch. 1. 18.

15. *Required to differentiate a product of two functions of different variables.*

$$\left.\begin{array}{l} \text{Let } u = \text{F}x, \text{U} = \text{F}(x+h) \\ v = fy, \text{V} = f(y+k) \end{array}\right\}, \text{therefore} \left\{\begin{array}{l} \text{U} = u + ph + qh^2 + \&c. \\ \text{V} = v + p'k + q'k^2 + \&c. \end{array}\right.$$

$$\begin{aligned} \text{and UV} = uv &+ pvh + qvh^2 + \&c. \\ &+ p'uk + pp'hk + \&c. \\ &+ q'uk^2 + \&c. \\ &+ \&c.; \end{aligned}$$

$$\begin{aligned} \text{therefore } \frac{\text{UV} - uv}{h} = pv + p'u\frac{k}{h} &+ qvh + \&c. \\ &+ pp'k + \&c. \\ &+ q'u\frac{k^2}{h} + \&c. \\ &+ \&c.; \end{aligned}$$

and assuming that x and y may be considered as dependent quantities, and taking the ratios in their limit, we have

$$\frac{d.uv}{dx} = pv + p'u\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dy}\frac{dy}{dx}, \text{ therefore}$$

$$d.uv = v\frac{du}{dx}dx + u\frac{dv}{dy}\frac{dy}{dx}dx = vdu + udv.$$

16. *Required to differentiate the quotient of two functions of different variables.*

Inc. $\frac{u}{v} = \frac{\text{U}}{v} - \frac{u}{v} = \frac{u + ph + qh^2 + \&c.}{v + p'k + q'k^2 + \&c.} - \frac{u}{v} = \frac{pvh - p'uk}{v(v + p'k)}$ + &c.

$$\text{therefore } \frac{\text{inc.}\,\frac{u}{v}}{h} = \frac{pv - p'u\frac{k}{h}}{v(v+p'k)};$$

$$\text{and } \frac{d.\frac{u}{v}}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dy}\frac{dy}{dx}}{v^2},$$

$$\text{or } d.\frac{u}{v} = \frac{1}{v}\frac{du}{dx}dx - \frac{u}{v^2}\cdot\frac{dv}{dy}\frac{dy}{dx}dx$$

$$= \frac{du}{v} - \frac{udv}{v^2}.$$

17. *The demonstration of these rules may easily be extended to any number of functions containing any number of variables.*

Thus let $\left.\begin{array}{l} u = \mathrm{F}(x, y) \\ v = fz \end{array}\right\}$ and let it be required to show that $d.uv = udv + vdu$.

$\left.\begin{array}{l} \mathrm{U} = \mathrm{F}(x+h, y+k) \\ \mathrm{V} = f(z+l) \end{array}\right\}$ then (Arts. 6 & 1) $\left\{\begin{array}{l} \mathrm{U} = u + ph + p'k + \&c. \\ \mathrm{V} = v + p''l + \&c. \end{array}\right\}$

therefore $\dfrac{\mathrm{UV} - uv}{h} = vp + vp'\dfrac{k}{h} + up''\dfrac{l}{h} + \&c.$; hence

$$\frac{d.uv}{dx} = v\frac{du}{dx} + v\frac{du}{dy}\frac{dy}{dx} + u\frac{dv}{dz}\frac{dz}{dx}, \text{ and}$$

$$d.uv = v\left\{\frac{du}{dx}dx + \frac{du}{dy}\frac{dy}{dx}dx\right\} + u\frac{dv}{dz}\frac{dz}{dx}dx$$

$$= \text{(Arts. 9 and 8) } vdu + udv.$$

CHAPTER IV.

On the different orders of Fluxions; the developement of Functions into series; and successive differentiations.

1. If a variable increases uniformly, its increment being a constant quantity, admits not of a fluxion (1.9); but if it increases with an accelerated or a diminished velocity, its increment is not constant, and it must have a fluxion which is to be calculated upon the same principles and by the same rules as the fluxion of the variable itself. If the increment of the increment be not constant there will be a third fluxion; and thus there arise different orders of fluxions, which should be represented by appropriate notation.

The principal variable is in general supposed to flow uniformly, and the fluxions of the function are calculated on that supposition. Thus take x^2, the function of x; then if x increase by the numbers 1, 3, 5, 7, ····, its increment being constant, it cannot have a second fluxion; but x^2 increases according to the numbers 1, 9, 25, 49, ····, and its successive increments are 8, 16, 24, ····, and consequently x^2 has a second fluxion which is positive (1.9). Since on the same supposition the increment of the increment of x^2 is a constant quantity, it does not admit of a third fluxion.

2. *The fluxional coefficients of a function of one variable.*

Let $u=fx$, then it has been shown (1.33)

that $\dfrac{du}{dx} = \phi x =$, by substitution, p.

Now, p containing the variable x, it may be differentiated; let $dp=qdx$ where q is a function of x; hence, we have $d.\dfrac{du}{dx} = dp = qdx$, or if dx be supposed a constant quantity, dividing by dx, we have $\dfrac{d.du}{dx^2} = q$. By differentiating successively on the same supposition, and dividing by dx, we

shall have $\frac{d.d.du}{dx^3} = r$, $\frac{d.d.d.du}{dx^4} = s$, &c. = &c. where r, s, &c. are functions of x. The operation may be continued till x ceases to enter into the function.

According to the notation explained in Ch. 1. 3., $d.du$, or the second fluxion of u, is denoted by d^2u; the third, fourth, ... nth fluxions of u are denoted by d^3u, $d^4u \cdots\cdots d^nu$. These must be distinguished from du^2, du^3, $du^4 \cdots\cdots du^n$, which represent different powers of du.

We must also distinguish between dx^n and $d.x^n$, the latter of which $= nx^{n-1}\,dx$; so also $d^2\,y^2$ and $d^2.\,y^2$ are different, the first being the square of the second fluxion, and the other the second fluxion of the square.

In Newton's notation, which till within these few years has been adopted by all the English mathematicians, the 2d, 3d, 4th .... fluxions are denoted by $\ddot{u}$, $\dddot{u}$, $\ddddot{u}$, ... or by $u^{2\cdot}$, $u^{3\cdot}$, $u^{4\cdot} \ldots\ldots u^{n\cdot}$; and the different powers of the first fluxion by $u^{\cdot 2}$, $u^{\cdot 3}$, $u^{\cdot 4} \ldots\ldots u^{\cdot n}$.

The quantities p, q, r, &c. or their equals $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, $\frac{d^3u}{dx^3} \cdots\cdots \frac{d^nu}{dx^n}$ are called the 1st, 2d, 3d .... nth *fluxional coefficients* of the function $u = fx$.

If u is a function of more than one variable, its different fluxional coefficients will depend upon the suppositions that are made with respect to the variables.

Ex. 1. Let $u = ax^n$;

$$\text{then } p = \frac{du}{dx} = nax^{n-1}$$

$$q = \frac{d^2u}{dx^2} = \frac{dp}{dx} = n.(n-1)\,ax^{n-2}$$

$$r = \frac{d^3u}{dx^3} = \frac{d^2p}{dx^2} \text{ or } = \frac{dq}{dx} = n.(n-1)(n-2)ax^{n-3}$$

&c. = &c.

or the fluxional coefficients of u are equal to nax^{n-1}, $n.(n-1)\,ax^{n-2}$, $n.(n-1)\,(n-2)ax^{n-3}$, &c.

If n is a positive integer, this example admits of only n fluxions, for the nth coefficient is $\frac{d^nu}{dx^n} = n.(n-1)\,(n-2)\cdots\cdots 3.2.1 \times a$, which is a constant quantity.

Ex. 2. Required the nth fluxion of pq, p and q being functions of the same variable x.

Let $u = pq \therefore du = pdq + qdp \therefore$

$\frac{du}{dx} = p\frac{dq}{dx} + q\frac{dp}{dx} \therefore$ differentiating and dividing by dx,

$\frac{d^2u}{dx^2} = p\frac{d^2q}{dx^2} + \frac{2dp}{dx}\frac{dq}{dx} + \frac{qd^2p}{dx^2}$ or . . . .

$d^2u = pd^2q + 2dp\ dq + qd^2.p.$

Similarly it may be shown that $d^3u = pd^3q + 3d^2q\ dp + 3d^2p\ dq + qd^3p$; and by the method of induction it may be demonstrated that

$$d^nu = pd^nq + nd^{n-1}q\ dp + n.\frac{n-1}{2}\ d^{n-2}q\ d^2p + \&c.$$

Ex. 3. $du = ydx - xdy$

$\therefore d^2u = dxdy + yd^2x - dxdy - xd^2y$

$= yd^2x - xd^2y = -xd^2y$ if dx be supposed constant;

or $= yd^2x$ if dy be constant.

Ex. 4. $u = \frac{ydx}{dy}$

$$\therefore du = dx + \frac{yd^2x}{dy} - \frac{ydxd^2y}{dy^2}.$$

Ex. 5. Required to differentiate $\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{-dxd^2y} = d^2u.$

Here we may not suppose dy constant, because d^2y enters into the function; and making dx constant, we have

$$d^3u = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx}\ \frac{d^3y}{d^2y^2} - 3(dx^2 + dy^2)^{\frac{1}{2}}\ \frac{dy}{dx}.$$

3. *The partial fluxional coefficients of a function of the sum of two variables are equal to one another.*

Let $u = f(x+y)$, then shall $\frac{du}{dx} = \frac{du}{dy}, \frac{d^2u}{dx^2} = \frac{d^2u}{dy^2}, \frac{d^3u}{dx^3} = \frac{d^3u}{dy^3}$, &c. = &c.

For in the expansion of $u = f(x+y+h)$, it can make no difference in the result whether we suppose h to be the increment of x or of y, hence the coefficient of the second term is the same on either supposition, or $\frac{du}{dx} = \frac{du}{dy}$.

Again, since $\frac{du}{dx}$ and $\frac{du}{dy}$ are equal functions of $(x+y)$, it follows as before that their partial fluxional coefficients are equal, or $\frac{d^2u}{dx^2}=\frac{d^2u}{dy^2}$, &c. = &c.

Taylor's Theorem.

4. *If* $u=fx$, *then shall* $f(x+h)=u+\frac{du}{dx}.\frac{h}{1}+\frac{d^2u}{dx^2}.\frac{h^2}{1.2}+\frac{d^3u}{dx^3}.\frac{h^3}{1.2.3}+$ &c.

For (Ch. 3. Theor. 1.) $f(x+h)=u+ph+qh^2+rh^3+$ &c. where $p, q, r, \cdots$ &c. are functions of x independent of h; but (Art. 3.) $\frac{d.f(x+h)}{dx}=\frac{d.f(x+h)}{dh}$ i. e. differentiating,

$$\frac{du}{dx}+h\frac{dp}{dx}+h^2\frac{dq}{dx}+h^3\frac{dr}{dx}+\text{\&c.}=p+2qh+3rh^2+4sh^4+\text{\&c.}$$

& equating terms which contain the same powers of h, (Alg. 346)

$$\frac{du}{dx}=p,\ \frac{dp}{dx}=2q,\ \frac{dq}{dx}=3r,\ \frac{dr}{dx}=4s,\ \text{\&c.}=\text{\&c.};\ \text{hence } p=\frac{du}{dx};$$

$$q=\tfrac{1}{2}\frac{dp}{dx}=\tfrac{1}{2}\frac{d^2u}{dx^2};\ r=\tfrac{1}{3}\frac{dq}{dx}=\frac{1}{1.2.3}\frac{d^3u}{dx^3};\ s=\frac{1}{1.2.3.4}\frac{d^4u}{dx^4};$$

&c. = &c. and $f(x+h)=u+\frac{du}{dx}.\frac{h}{1}+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+$ &c.*

For the sake of conciseness, we shall frequently substitute p, q, r, &c. for the 1st, 2d, 3d, ... fluxional coefficients of the function.

5.† *Taylor's Theorem may be deduced from first principles without the aid of Theor.* 1. *Ch.* 3.

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* The demonstration of Theor. 1. Ch. 3. upon which this depends has been improved by Poisson, who shows that the exponents of h must ascend by the numbers 1, 2, 3, &c.

† It appears from Vol. 2. Prop. 10, of the Principia, that Sir I. Newton was aware that the increment of a function may be expanded in a series of the ascending powers of the increment of its variable. It is probable that his own binomial theorem led him to this conclusion, for it is by means of that theorem that he has obtained the value of the coëfficients in several examples. But it is by no means certain that he was aware of the law by which they can be derived from each other in succession; and as this was first published by Dr. Brook Taylor, in his *Methodus Incrementorum*, the Theorem has been always called Taylor's.

For, since u is a function of x, we may assume fx or $u = a + bx^m + cx^n + \&c.$ where a, b, c, &c. are constant quantities; therefore

$$f(x+h) = a + b.(x+h)^m + c.(x+h)^n + \&c.$$

$$\left.\begin{aligned} &= a + b\,x^m + c\,x^n + \&c. \\ &\quad + (mbx^{m-1} + nc\,x^{n-1} + \&c.)\frac{h}{1} \\ &\quad + (m.(m-1)bx^{m-2} + n.(n-1)cx^{n-2} + \&c.)\frac{h^2}{1.2} \\ &\quad + \&c. \end{aligned}\right\}$$

$$= u + \frac{du}{dx}\cdot\frac{h}{1} + \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} + \&c.$$

Cor. 1. Let h be negative, and denote the preceding value of u by ${}_{\prime}u$; then by the theorem we have

$${}_{\prime}u = u - p.\frac{h}{1} + q.\frac{h^2}{1.2} - r.\frac{h^3}{1.2.3} + \&c.$$

Cor. 2. Since dx may be assumed of any value (1.7), suppose $dx = h$, then

$$\left.\begin{aligned} f(x+h) \text{ or } \text{υ} &= u + \frac{du}{1} + \frac{d^2u}{1.2} + \frac{d^3u}{1.2.3} + \&c. \\ \text{and } {}_{\prime}u &= u - \frac{du}{1} + \frac{d^2u}{1.2} - \frac{d^3u}{1.2.3} + \&c. \end{aligned}\right\} \text{which series}$$

are therefore functions of x + inc. x and of x − inc. x respectively.

6. *Maclaurin's Theorem.*

$fx = f. + \frac{x}{1}f.' + \frac{x^2}{1.2}f.'' + \frac{x^3}{1.2.3}f.''' + \&c.$ *where* $f.$, $f.'$, $f.''$, $f.'''$, &c. *represent the values of* u, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, $\frac{d^3u}{dx^3}$, *&c. when* $x = 0$.

Let $fx = a + bx + cx^2 + cx^3 + \&c.$, then we have

	∴	or
$fx = a + bx + cx^2 + ex^3 + \&c.$	$f. = a$	$a = f.$
$f'x = b + 2cx + 3ex^2 + \&c.$	$f.' = b$	$b = f.'$
$f''x = 2c + 2.3ex + \&c.$	$f.'' = 2c$	$c = \frac{f.''}{2}$
$f'''x = 2.3e + \&c.$	$f.''' = 2.3e$	$e = \frac{f.'''}{2.3}$
&c. = &c.	&c. = &c.	&c. = &c.

Hence $fx = f. + \frac{x}{1} f.' + \frac{x^2}{1.2} f.'' + \frac{x^3}{1.2.3} f.''' + \&c.$

Maclaurin's Fluxions, Vol. ii. Ch. 2. Art. 748.

7. *Required to deduce Maclaurin's Theorem from Taylor's.*

$$f(x+h) = u + \frac{du}{dx} \cdot \frac{h}{1} + \frac{d^2u}{dx^2} \cdot \frac{h^2}{1.2} + \frac{d^3u}{dx^3} \cdot \frac{h^3}{1.2.3} + \&c.$$

or $f(x+h) = fx + f'x . \frac{h}{1} + f''x . \frac{h^2}{1.2} + f'''x . \frac{h^3}{1.2.3} + \&c.$

where x and h are any indeterminate quantities.

Suppose $x=0$, and the equation becomes

$$fh = f. + \frac{h}{1} f.' + \frac{h^2}{1.2} f.'' + \frac{h^3}{1.2.3} f.''' + \&c.$$

or, replacing h by x,

$$fx = f. + \frac{x}{1} f.' + \frac{x^2}{1.2} f.'' + \frac{x^3}{1.2.3} f.''' + \&c.$$

As this demonstration contains a *fallacia suppositionis*, we shall subjoin

8. *Lagrange's Demonstration.*

In Taylor's Theorem diminish x by h, then we have

$$fx = f(x-h) + f'(x-h) . \frac{h}{1} + f''(x-h) . \frac{h^2}{1.2} . + \&c.$$

Substitute $h = xz$ where z may be any quantity whatever, and we have

$$fx = f(x - xz) + f'(x - xz) \frac{h}{1} + f''(x - xz) \frac{h^2}{1.2} + \&c.$$

Suppose $z=1$ or $h=x$, and there results

$fx = f. + \frac{x}{1} f.' + \frac{x^2}{1.2} f.'' + \&c.$ (Fonct. Anal. Ch. 6.)

Cor. If any of the coefficients are infinite, the function cannot be developed in a series ascending by the positive integral powers of x.

Thus, if $fx = lx$, $f'x = \frac{1}{x}$, $f''x = -\frac{1}{x^2}$, &c. = &c.

and $f. = \infty$, $f.' = \infty$, &c. = &c. and lx cannot be developed in a series of the form $A + Bx + Cx^2 + \&c.$

Examples.

(1.) Required to deduce the binomial theorem from Maclaurin's.

$$\left.\begin{array}{ll|ll} fx &= (1+n)^n & f. &= 1 \\ f^{\prime}x &= n.(1+x)^{n-1} & f.^{\prime} &= n \\ f^{\prime\prime}x &= n.(n-1)\,(1+x)^{n-2} & f.^{\prime\prime} &= n.(n-1) \\ f^{\prime\prime\prime}x &= n.(n-1)\,(n-2)\,(1+x)^{n-3} & f.^{\prime\prime\prime} &= n.(n-1)\,(n-2) \\ \&c. &= \&c. & \&c. &= \&c. \end{array}\right\}$$

$$\therefore (1+x)^n = 1 + \frac{x}{1}.n + \frac{x^2}{1.2}.\, n.(n-1) + \&c.$$

(2.) To expand a^x.

$$\left.\begin{array}{ll|ll} fx &= a^x & f. &= 1 \\ f^{\prime}x &= \text{A}a^x & f.^{\prime} &= \text{A} \\ f^{\prime\prime}x &= \text{A}^2a^x & f.^{\prime\prime} &= \text{A}^2 \\ f^{\prime\prime\prime}x &= \text{A}^3a^x & f.^{\prime\prime\prime} &= \text{A}^3 \\ \&c. &= \&c. & \&c. &= \&c. \end{array}\right\} \therefore\ a^x = 1 + \frac{x}{1}.\,\text{A} + \frac{x^2}{1.2}\,\text{A}^2 + \&c.$$

(3.) To expand $l(a+x)$.

$$\begin{array}{ll|ll} fx &= l(a+x) & f. &= la \\ f^{\prime}x &= \dfrac{1}{a+x} = (a+x)^{-1} & f.^{\prime} &= \dfrac{1}{a} \\ f^{\prime\prime}x &= -\dfrac{1}{(a+x)^2} = -(a-x)^{-2} & f.^{\prime\prime} &= -\dfrac{1}{a^2} \\ f^{\prime\prime\prime}x &= 2.(a+x)^{-3} & f.^{\prime\prime\prime} &= +\dfrac{2}{a^3} \\ f^{\text{iv}}x &= -2.3(a+x)^{-4} & f.^{\text{iv}} &= \dfrac{-2.3}{a^4} \\ \&c. &= \&c. & \&c. &= \&c. \end{array}$$

therefore $l(a+x) = la + \dfrac{x}{a} - \dfrac{x^2}{2a^2} + \dfrac{x^3}{3a^3} - \dfrac{x^4}{4a^4} + \&c.$

If in this function we suppose $a=0$, we have

$lx = -\infty + \dfrac{x}{0} - \dfrac{x^2}{0} + \&c.$, or the series fails when we assign this particular value to a.

(4.) Required the sine in terms of its arc.

$$\left.\begin{array}{l|l} f\ x=\sin.x & f. = 0 \\ f'\ x=\cos.x & f.' = 1 \\ f''\ x=-\sin.x & f.'' = 0 \\ f'''x=-\cos.x & f.''' = -1 \\ f^{\text{iv}}x=\sin.x & f.^{\text{iv}} = -0 \\ \&\text{c.} =\&\text{c.} & f.^{\text{v}} = 1 \end{array}\right\} \therefore \sin.x=\frac{x}{1}-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5} - \&\text{c.}$$

Similarly it may be shown that

$$\cos.x=1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\&\text{c.}$$

$$\text{and } \tan.x=x+\frac{2x^3}{1.2.3}+\frac{16x^5}{1.2.3.4.5}-\&\text{c.}$$

(5.) The developement of tan.x in the last example does not give the law of the series; let it be required then to investigate this law.

$$\text{Tan.}x=\frac{\sin.x}{\cos.x}=\frac{\dfrac{x}{1}-\dfrac{x^3}{1.2.3}+\dfrac{x^5}{1.2.3.4.5}-\&\text{c.}}{1-\dfrac{x^2}{2}+\dfrac{x^4}{1.2.3.4}-\&\text{c.}}.$$

Assume therefore

$\tan.x=A_1x+A_3x^3+A_5x^5+\dots+A_{2n+1}\,x^{2n+1}+\&\text{c.}$; hence

$$x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\&\text{c.}$$

$$\begin{aligned}&=A_1x+A_3x^3+A_5x^5+\dots+A_{2n+1}\,x^{2n+1}+\dots\\ &\quad-\frac{A_1x^3}{1.2}-\frac{A_3x^5}{1.2}-\dots-\frac{A_{2n-1}\,x^{2n+1}}{1.2}-\dots\\ &\quad+\frac{A_1x^5}{1.2.3.4}+\dots+\frac{A_{2n-3}\,x^{2n+1}}{1.2.3.4}+\dots\\ &\quad-\frac{A_{2n-5}\,x^{2n-1}}{1.2.3.4.5.6}-\dots\\ &\quad+\&\text{c.}\end{aligned}$$

Hence, equating the coefficients of like terms (Alg. 346), we have $\therefore$

$$\begin{array}{r|l}
A_1 = 1 & A_1 = 1 \\
A_3 - \dfrac{A_1}{1.2} = -\dfrac{1}{1.2.3} & A_3 = \dfrac{A_1}{1.2} - \dfrac{1}{1.2.3} = \dfrac{2}{1.2.3} \\
A_5 - \dfrac{A_3}{1.2} + \dfrac{A_1}{1.2.3.4} = \dfrac{1}{1.2.3.4.5} & A_5 = \dfrac{A_3}{1.2} - \dfrac{A_1}{1.2.3.4} + \dfrac{1}{1.2.3.4.5} \\
\&c. = \&c. & \quad = \dfrac{16}{1.2.3.4.5} \\
 & \&c. = \&c.
\end{array}$$

Generally $A_{2n+1} = \dfrac{A_{2n-1}}{1.2} - \dfrac{A_{2n-3}}{1.2.3.4} + \ldots \dfrac{\pm A_1}{1.2\ldots 2n} \ldots\ldots$

$$\mp \frac{1}{1.2\ldots\ldots(2n+1)}.$$

Since all the trigonometrical lines may be expressed in terms of the sine and of the cosine, it is obvious that they all may be developed in terms of the are by means of this method of indeterminate coefficients; thus, take for the next example

(6.) $u = \cot. x$.

In this case both Maclaurin's and Taylor's Theorems fail to give the required developement; for the first term and the coefficients of all the succeeding terms become infinite.

But since $\cot. x = \dfrac{\cos. x}{\sin. x} = \dfrac{1 - \dfrac{x^2}{2} + \dfrac{x^4}{1.2.3.4} - \&c.}{\dfrac{x}{1} - \dfrac{x^3}{1.2.3} + \dfrac{x^5}{1.2.3.4.5} - \&c.}$; assume

$$\cot. x = \frac{1}{x} + A_2 x + A_4 x^3 + \ldots + A_{2n} x^{2n-1} + \&c. \therefore$$

$$1 - \frac{x^2}{2} + \frac{x^4}{1.2.3.4} - \&c.$$

$$\begin{aligned}
= 1 &+ A_2 x^2 + A_4 x^4 + \ldots\ldots + A_{2n} x^{2n} + \&c. \\
&- \frac{x^2}{1.2.3} - \frac{A_2 x^4}{1.2.3} - \ldots\ldots - \frac{A_{2n-2}\, x^{2n}}{1.2.3} - \&c. \\
&+ \frac{x^4}{1.2.3.4.5} + \ldots + \frac{A_{2n-4}\, x^{2n}}{1.2.3.4.5} + \&c. \\
&- \frac{A_{2n-6}\, x^{2n}}{1.2.3.4.5.6.7} - \&c. \\
&+ \&c.
\end{aligned}$$

Hence $\qquad\qquad\qquad\qquad \therefore$

$$-\frac{1}{2} = A_2 - \frac{1}{1.2.3} \qquad \Bigg| \qquad A_2 = -\frac{1}{3}$$

$$+\frac{1}{1.2.3.4} = A_4 - \frac{A_2}{1.2.3} + \frac{1}{1.2.3.4.5} \qquad \Bigg| \qquad A_4 = -\frac{1}{3^2.5}$$

&c. = &c. $\qquad\qquad$ &c. = &c.

and generally $A_{2n} = \frac{A_{2n-}}{1.2.3} - \frac{A_{2n-4}}{1.2.3.4.5} + \dots \pm \frac{A_2}{1.2\dots(2n-1)}$

$$\mp \frac{1}{1.2\dots(2n-1).(2n+1)}.$$

(7.) $u = \sin.^{-1}x$.

$fx = \sin.^{-1}x$

$f'x = (1-x^2)^{-\frac{1}{2}}$

$f''x = x.(1-x^2)^{-\frac{3}{2}}$

$f'''x = (1-x^2)^{-\frac{3}{2}} + 3x^2(1-x^2)^{-\frac{5}{2}}$

&c. = &c.

$$\therefore \sin.^{-1}x = \frac{x}{1} + \frac{1^2.x^3}{1.2.3} + \frac{1^2.3^2.x^5}{1.2.3.4.5} + \text{\&c.}$$

$$\text{or } u = \sin.u + \frac{\sin.^3u}{2.3} + \frac{3\sin.^5u}{2.4.5} + \text{\&c.}$$

$$\text{Similarly } \cos.^{-1}x = \frac{\pi}{2} - \left(\frac{x}{1} - \frac{x^3}{1.2.3} + \frac{3x^5}{2.4.5} - \text{\&c.}\right)$$

$$\text{and } \tan.^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{\&c.}$$

Cor. Suppose $u = 30°$, then $\sin.u = \frac{1}{2}$ $\therefore$

$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{8} \times \frac{1}{2.3} + \frac{1}{32} \times \frac{3}{2.4.5} + \text{\&c.}$; from which an approximate value of y may be obtained.

(8.) Required the terminus generalis of the developement of $u = \sin.^{-1}x$.

$$\frac{du}{dx} = (1-x^2)^{-\frac{1}{2}}$$

$$= 1 + \frac{\frac{1}{2}x^2}{1} + \frac{\frac{1}{2}.\frac{3}{2}x^4}{1.2} + \frac{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}x^6}{1.2.3} + \dots + \frac{\frac{1}{2}.\frac{3}{2}\dots\frac{n-1}{2}.x^n}{1.2\dots\frac{n}{2}} + \text{\&c.}$$

$$=1+\frac{1}{2}x^2+\frac{1.3}{2.4}x^4+\frac{1.3.5}{2.4.6}x^6+\ldots\ldots\frac{1.3.5\ldots(n-1)}{2.4.6\ldots n}x^n+\&c.$$

where n is necessarily an even number. Assume then

$$\frac{du}{dx}=1+\text{A}x^2+\text{B}x^4+\ldots\text{N}x^n+\text{M}x^{n+2}+\&c. \text{ where}$$

$$\text{N}=\frac{1.3.5\ldots\ldots(n-1)}{2.4.6\ldots\ldots n}. \quad \therefore$$

$$\frac{d^2u}{dx^2}=2\text{A}x+4\text{B}x^3+\ldots\ldots+n\text{N}x^{n-1}+(n+2)\text{M}x^{n+1}+\&c.$$

$$\frac{d^3u}{dx^3}=2\text{A}+3.4\text{B}x^2+\ldots+n.(n-1)\text{N}x^{n-2}+(n+1)(n+2)\text{M}x^n+\&c.$$

$$\ldots=\ldots\ldots\ldots$$

$$\frac{d^{n+1}u}{dx^{n+1}}=n.(n-1)\ldots\ldots2.1\,\text{N}+(n+2)(n+1)\ldots\ldots3.2\,\text{M}x^2+\&c.$$

$\therefore\left(\frac{d^{n+1}u}{dx^{n+1}}\right)=n.(n-1)\ldots\ldots2.1.\ \text{N}$, and consequently the terminus generalis, which by the Theorem . . . .

$$=\left(\frac{d^{n+1}u}{dx^{n+1}}\right).\ \frac{x^{n+1}}{1.2\ldots\ldots n.(n+1)},=\frac{\text{N}}{n+1}.x^{n+1}$$

$$=\frac{1.3.5\ldots\ldots n-1}{2.4.6\ldots\ldots n.(n+1)}.\ x^{n+1}.$$

It may be observed that the fluxional coefficients of the even powers vanish as they ought when $x=0$. (Lacroix, tome 1, p. 253.)

Similarly we may find the terminus generalis of $u=\cos.^{-1}x$, $u=\tan.^{-1}x$, &c.

(9.) Required to develope $\sin.x$ by means of Taylor's Theorem.

$u=\sin.x \;\therefore\; \text{U}=\sin.(x+h) \;\therefore$

$$\left.\begin{array}{l} p=\cos.x \\ q=-\sin.x \\ r=-\cos.x \\ s=\sin.x \\ \&c.=\&c. \end{array}\right\}\therefore\ \sin.(x+h)=$$

$$\sin.x+\cos.x\frac{h}{1}-\sin.x\frac{h^2}{1.2}-\cos.x\frac{h^3}{1.2.3}+\&c.;$$

in which, suppose $x=0$ and replace h by x, and there results

$$\sin.x=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\&c. \text{ as in Ex. 4.}$$

Similarly $\cos.(x+h)=\cos.x - \sin.x\frac{h}{1} - \cos.x\frac{h^2}{1.2} + \ldots$ $\sin.x\frac{h^3}{1.2.3} - \&c.$; and $\cos.x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.$

(10.) To expand a multinomial of the form $(a+bx+cx^2+ex^3+\&c.)^n$.

$$
\begin{aligned}
fx &= (a+bx+cx^2+\&c.)^n \\
f'x &= n.(a+bx+cx^2+\&c.)^{n-1}\{b+2cx+3ex^2+\&c.\} \\
f''x &= n.(n-1)\,(a+bx+cx^2+\&c.)^{n-2}\{b+2cx+\&c.\}^2 \ldots \\
&\quad + n.(a+bx+\&c.)^{n-1}\{2c+2.3ex+\&c.\} \\
f'''x &= n.(n-1)\,(n-2).(a+bx+\&c.)^{n-3}\{b+2cx+\&c.\}^3 \ldots \\
&\quad + 2n.(n-1)\,(a+bx+\&c.)^{n-2}(b+2cx+\&c.) \ldots \\
&\quad \{2c+3ex+\&c.\} + n.(n-1)\,(a+bx+\&c.)^{n-2} \ldots \\
&\quad \{2c+2.3ex+\&c.\}\,\{b+2cx+\&c.\} \ldots \\
&\quad + n.(a+bx+\&c.)^{n-1}\{2.3e+\&c.\}
\end{aligned}
$$

&c. = &c; hence we have

$$
\begin{aligned}
f. &= a^n \\
f.' &= na^{n-1}b \\
f.'' &= n.(n-1)\,a^{n-2}b^2 + na^{n-1}.2c \\
f.''' &= n.(n-1)\,(n-2)\,a^{n-3}b^3 + 4n.(n-1)a^{n-2}bc \ldots \\
&\quad + 2n.(n-1)\,a^{n-2}bc + 2.3na^{n-1}e
\end{aligned}
$$

&c. = &c.; hence by the theorem

$$
fx = a^n + \frac{n}{1}a^{n-1}bx + \left.\begin{matrix} \frac{n.(n-1)}{1.2}a^{n-2}b^2 \\ + \frac{n}{1}a^{n-1}c \end{matrix}\right\} x^2 + \left.\begin{matrix} \frac{n.(n-1)(n-2)}{1.2.3}a^{n-3}b^3 \\ + \frac{n.(n-1)}{1.1}a^{n-2}bc \\ + \frac{n}{1}a^{n-1}e \end{matrix}\right\} x^3 + \&c.
$$

This expansion is De Moivre's. For the method of obtaining the terminus generalis see his works, and Lagrange's Calcul des Fonctions, Leçon 4ieme.

Similarly it may be shown that

$$e^{a+bx+cx^2+kx^3+\&c.}=e^a+e^a bx$$
$$+\left\{e^a c+\frac{e^a b^2}{1.2}\right\}x^2+\left\{e^a k+\frac{e^a 2bc}{1.2}+\frac{e^a b^3}{1.2.3}\right\}x^3+\&c.$$
$$l(a+bx+cx^2+ex^3+\&c.)=la+a^{-1}b.x$$
$$+\left\{a^{-1}c-\frac{a^{-2}b^2}{1.2}\right\}x^2+\left\{a^{-1}e-\frac{a^{-2}\,2bc}{1.2}+\frac{a^{-3}\,b^3}{1.3}\right\}x^3+\&c.$$

and $\sin.(a+bx+cx^2+ex^3+\&c.)$

$$=\sin.a+b\cos.a.\,x-(b\sin.a-2c\cos.a)\frac{x^2}{1.2}-\&c.$$

(11.) Let $my^3-xy=m$; required to develope y in a series ascending by the powers of x.

$(3my^2-x)dy-ydx=0$ ∴

$$p=\frac{y}{3my^2-x}$$
$$q=\frac{p}{3my^2-x}-\frac{2.3\,my^2p-y}{(3my^2-x)^2}=\frac{y-3\,my^2p-xp}{(3my^2-x)^2}$$

&c. = &c.

But when $x=0$, $y=1$; ∴ $(y)=1$, $(p)=\frac{1}{3m}$, $(q)=0$, ...

∴ $y=1+\frac{x}{3m}-\frac{x^3}{3^4m^3}+\frac{x^4}{3^5m^4}-$ &c. ...., which is one of the three values of y in terms of x. (Alg. 266.)

9. *The cases in which Maclaurin's Theorem fails may frequently be solved by substituting* $u=x^kz$, k *being assumed such that* (z), $\left(\frac{dz}{dx}\right)$, $\left(\frac{d^2z}{dx^2}\right)$, ... *may not be infinite.*

Ex. 1. $u=\cot.x$

Assume $\cot.x=x^kz$ or $z=\frac{\cot.x}{x^k}$.

In this case k may not be a positive number, for $x=0$ would make $z=\infty$; assume then $k=-1$, or

$$z=x.\cot.x=x.\frac{\cos.x}{\sin.x}=\frac{1-\frac{x^2}{2}+\frac{x^4}{1.2.3.4}-\&c.}{1-\frac{x^2}{1.2.3}+\frac{x^4}{1.2.3.4.5}-\&c.}.$$

By differentiating successively we have

$$\begin{array}{l|l} (z) \quad =1 & \therefore \\ \left(\dfrac{dz}{dx}\right)=0 & z=1-\dfrac{x^3}{3}+\dfrac{x^4}{3^2.5}-\&c. \\ \left(\dfrac{d^2z}{dx^2}\right)=-\tfrac{2}{3} & \& \ u=\dfrac{z}{x}=x^{-1}-\dfrac{x}{3}-\dfrac{x^3}{3^2.5}-\dfrac{2x^5}{3^2.5.7}-\dfrac{x^7}{3^2.5^2.7}-\&c. \\ \&c.=\&c. & \end{array}$$

Ex. 2. Let $my^3-x^3y-mx^3=0$; required to develope y in a series ascending by the powers of x.

Substitute $y=x^kz$, $\therefore mz^3x^{3k}-zx^{k+3}-mx^3=0$, and dividing by x^3 that the last term may be a number, we have, $mz^3x^{3k-3}-zx^k-m=0$.

Assume $k=1$, then the equation becomes $mz^3-zx-m=0$, and consequently (8. Ex. 11)

$$z=1+\frac{x}{3m}-\frac{x^3}{3^4m^3}+\frac{x^4}{3^5m^4}-\&c.$$

$$\therefore\ y=xz=x+\frac{x^2}{3m}-\frac{x^4}{3^4m^3}+\frac{x^5}{3^5m^4}-\&c.$$

Ex. 3. In the preceding example, required to develope y in a series ascending by the powers of m.

In this case we must consider x as a constant quantity; then by the theorem, if $p, q, r, \ldots$ represent $\frac{dy}{dm}, \frac{d^2y}{dm^2}, \frac{d^3y}{dm^3}, \ldots$; since $y=0$ when $m=0$, we have

$$p=-\frac{y^3-x^3}{3my^2-x^3}\quad \therefore (p)=-1$$

$$q=-\frac{3y^2p}{3my^2-x^3}+(y^3-x^3).\frac{3y^2+2.3myp}{(3my^2-x^3)^2}\quad \therefore (q)=0$$

$$\therefore (r)=-\frac{2.3yp^2}{3my^2-x^3}-\frac{x.^32.3\{2yp+mp^2\}}{(3my^2-x^3)^2}=0$$

$$\therefore (s)=-\frac{2.3p^3}{3my^2-x^3}-\frac{x^3.2.3.3p^2}{(3my^2-x^3)^2}=-\frac{6}{x^3}-\frac{18}{x^3}=-\frac{24}{x^3}$$

&c. = &c.

Hence, since $(y)=0$, the required series is

$$(\beta)y=-m-\frac{1}{x^3}.m^4-\frac{3}{x^6}m^7-\frac{12}{x^9}m^{10}\ldots\ldots$$

10. *By assuming different values for* k, y *may be developed into series which are frequently very dissimilar in their form.*

For *Ex.* 1. take Ex. 2. of the preceding article, which after substitution becomes $mz^3x^{3k-3}-zx^k-m=0$. (α).

1st. Assume $k=0$, then $mz^3x^{-3}-z-m=0$; substitute $u=x^{-3}$ and there results $mz^3u-z-m=0$.

To develope z in terms of u, we have

$$(z)=-m$$

$$p=-\frac{mz^3}{3mz^2u-1} \therefore (p)=-m^4$$

$$q=-\frac{3mz^2p}{3mz^2u-1}+\frac{mz.^3 3m\{2zup+z^2\}}{(3mz^2u-1)^2} \therefore (q)=-6m^7$$

&c. = &c. $\therefore z=-m-m^4u-3m^7u^2-$ &c.;

and replacing z and u by their values y and x^{-3}, there results

(β) $y=-m-\dfrac{m^4}{x^3}-\dfrac{3m^7}{x^6}-$ &c., which is the same series as that deduced 9. Ex. 3.

Next, since (α) may be reduced to $mz^3-zx^{3-2k}-mx^{3-3k}=0$, assume $k=\frac{3}{2}$, then (α) becomes $mz^3-z-mx^{-\frac{3}{2}}=0$; substitute $u=x^{-\frac{3}{2}}$ and there results $mz^3-z-mu=0$.

Now, to develope z in terms of u in this case, we have one value of $(z)=\sqrt{\dfrac{1}{m}}=\pm\dfrac{1}{m^{\frac{1}{2}}}$.

$$\text{Also } p=\frac{m}{3mz^2-1} \therefore (p)=\frac{m}{2}$$

$$q=-\frac{6m^2zp}{(3mz^2-1)^2} \therefore (q)=\mp\frac{3m^{\frac{5}{2}}}{4}$$

$$\text{\&c.}=\text{\&c.} \therefore z=\pm\frac{1}{m^{\frac{1}{2}}}+\frac{m}{2}u\mp\frac{3m^{\frac{5}{2}}}{8}u^2+\text{\&c.}$$

$$\text{and } y=x^{\frac{3}{2}}z=\pm\frac{x^{\frac{3}{2}}}{m^{\frac{1}{2}}}+\frac{m}{2}\mp\frac{3m^{\frac{5}{2}}}{8}x^{-\frac{3}{2}}+\text{\&c.}$$

Another value of (z) in this last developement $=0$; if we make use of this value in order to determine (p), (q), (r), ... we shall again fall upon the series (β).

It appears then that y in this example may be developed into four different series; but it can possess only three *Algebraick* values (Alg. 266), and the series (β) deduced, 9. Ex. 3, must be regarded as a mere analytical expression for y, which shows the form which y may assume when it is developed in a series ascending by the powers of m.

Ex. 2. $ay^4 - b^3xy - cx^4 = 0$.

Assuming $k=1$, the equation becomes, by substitution, $az^4 - b^3uz - c = 0$ and

$$y = xz = \frac{c^{\frac{1}{4}}}{a^{\frac{1}{4}}}x + \frac{b^3}{4a^{\frac{1}{2}}c^{\frac{1}{2}}x} - \frac{b^6}{1.2.4^2a^{\frac{3}{4}}c^{\frac{5}{4}}x^3} + \&c.$$

If k be assumed $= \frac{1}{3}$ or $= 3$, y may be developed in series ascending by the powers of x.

Ex. 3. $y^3 - 3axy + x^3 = 0$.

Assuming $k=1$, we have $y = -x - a - \dfrac{a^2}{x} - \dfrac{4a^3}{3x^2} - \&c.$ the coefficients containing the cube roots of unity.

This series is to be considered not as a real but as an *analytical* value of y: it is the same that we should obtain if we were to develope y in terms of a, considering x as a constant quantity.

The real values of y, or the three roots of the equation, will be found to be, by assuming $k=2$ and $k=\frac{1}{2}$,

$$y = \frac{x^2}{3a} + \frac{x^5}{3^4a^4} + \&c.$$

$$y = \pm\sqrt{3ax} - \frac{x^2}{6a} \mp \frac{x^3\sqrt{3ax}}{72a^3} - \frac{x}{2.3^4a^4} + \&c.$$

Ex. 4. $y^3 - 2xy^2 + x^2y - a^3 = 0$.

Substitute $y = x^kz \therefore x^{3k}z^3 - 2x^{2k+1}z^2 + x^{k+2}z - a^3 = 0$.

Assume $k=1$, then $z^3 - 2z^2 + z - a^3u = 0$ where $u = x^{-3}$.

$(z) = 0$, and $(z) = 1$; take $(z) = 0$.

$$p = \frac{a^3}{3z^2 - 4z + 1} \therefore (p) = a^3$$

$$q = -\frac{a^3(6zp - 4p)}{(3z^2 - 4z + 1)^2} \therefore (q) = 4a^6$$

$$\therefore (r) = -a^3((6z-4)q+6p^2)+2a^3(6zp-4p)^2$$
$$=4a^3q+26a^3p^2=42a^9$$
$$\&c.=\&c. \therefore y=x.z=\frac{a^3}{x^2}+\frac{2a^6}{x^5}+\frac{7a^9}{x^8}+\&c.$$

This is an *analytical* value of y; it may also be obtained by substituting $y=a^kz$, and assuming $k=-3$.

Other analytical values of y will be deduced as an example to Lagrange's Theorem.

11. In many of the preceding developements, the Theorems of Taylor and Maclaurin fail to give the terminus generalis of the series, or even its law. Euler has employed for this purpose with great success the method of Indeterminate Coefficients combined with the principle of differentiation. We shall subjoin but few instances, as the student may apply the same method to the examples of the preceding article.

Ex. 1. $u=\dfrac{(A+Bx+Cx^2+\&c.)^m}{(a+bx+cx^2+\&c.)^n}=\dfrac{v^m}{w^n}$ by substitution.

Hence $lu = mlv - nlw$, and differentiating,

$$\frac{1}{u}\frac{du}{dx}=\frac{m}{v}\frac{dv}{dx}-\frac{n}{w}\frac{dw}{dx} \text{ or } vw\frac{du}{dx}-muw\frac{dv}{dx}+nuv\frac{dw}{dx}=0\ (\alpha).$$

Assume $u=A_0+A_1x+A_2x^2+\&c.$;

then we have $A_0=\dfrac{A^m}{a^n}$; and to determine A_1, A_2, &c. we have by substitution in (α).

$$\begin{aligned}&\left\{A+Bx+Cx^2+\&c.\right\}\left\{a+bx+cx^2+\&c.\right\}\left\{A_1+2A_2x+\&c.\right\}\\ -m&\left\{A_0+A_1x+A_2x^2+\&c.\right\}\left\{a+bx+cx^2+\&c.\right\}\left\{B+2Cx+\&c.\right\}\\ +n&\left\{A_0+A_1x+A_2x^2+\&c.\right\}\left\{A+Bx+Cx^2+\&c.\right\}\left\{b+2cx+\&c.\right\}=0\end{aligned}$$

Hence (Alg. 347).

$$Aa.A_1+\left.\begin{matrix}nAb\\-mBa\end{matrix}\right\}A_0=0; \ldots\ldots\ldots\ldots\ldots\ldots$$

$$2Aa.A_2+\left.\begin{matrix}(n+1)Ab\\-(m-1)Ba\end{matrix}\right\}A_1+\left.\begin{matrix}2nAC\\+(n-m)Bb\\-2mCa\end{matrix}\right\}A_0=0;\ \&c.$$

By taking more terms of the series the law of their derivation will become more evident.

It is obvious that the success of this method depends entirely upon the degree of simplicity of the equation (α). If

there are radicals, they must be made to disappear by involution, and if the resulting equation should contain powers of u or of $\frac{du}{dx}$, they must be either eliminated or reduced by successive differentiations.

Ex. 2. $u=(x+\sqrt{x^2-1})^n$;

$\therefore lu=nl(x+\sqrt{x^2-1}) \therefore \frac{du}{u}=\frac{ndx}{\sqrt{x^2-1}}$

$\therefore (x^2-1)\,du^2=n^2u^2dx^2$; and to reduce this, differentiate again and there results, dividing by $2du\,dx^2$,

$$n^2u+(1-x^2)\frac{d^2u}{dx^2}-x\frac{du}{dx}=0 \quad (\alpha).$$

Assume $u=\text{A}_0+\text{A}_1x+\text{A}_2x^2+\text{A}_3x^3+\text{A}_4x^4+\&c.$

where $\text{A}_0=(\sqrt{-1})^n$;

$$\therefore \frac{du}{dx}=\text{A}_1+2\text{A}_2x+3\text{A}_3x^2+\&c. \ (\beta)$$

$$\text{and } \frac{d^2u}{dx^2}=2\text{A}_2+2.3\text{A}_3x+3.4\text{A}_4x^2+\&c.$$

Hence from equation (α) we have

$$\begin{aligned} &n^2\{\text{A}_0+\text{A}_1x+\text{A}_2x^2+\text{A}_3x^3+\text{A}_4x^4+\&c.\} \\ &+(1-x^2)\{2\text{A}_2+2.3\text{A}_3x+3.4\text{A}_4x^2+4.5\text{A}_5x^3+5.6\text{A}_6x^4+\&c.\} \\ &-x\{\text{A}_1+2\text{A}_2x+3\text{A}_3x^2+4\text{A}_4x^3+5\text{A}_5x^4+\&c.\} = 0. \end{aligned}$$

Hence $2\text{A}_2+n^2\text{A}_0=0$; $2.3\text{A}_3+(n^2-1)\text{A}_1=0$; $3.4\text{A}_4+(n^2-4)\text{A}_2=0$; $4.5\text{A}_5+(n^2-9)\text{A}_3=0$; $5.6\text{A}_6+(n^2-16)\text{A}_4=0$; &c. from which the law is sufficiently manifest; but since we have not A_1 in terms of A_0, it must be deduced independently; and from (β) we have $\text{A}_1=\left(\frac{du}{dx}\right)=n\left(\frac{u}{\sqrt{x^2-1}}\right)$ $=n.(\sqrt{-1})^{n-1}$; so that all the coefficients can be found in terms of $(\sqrt{-1})^n$ and of $(\sqrt{-1})^{n-1}$.

PRAXIS.

1. To deduce the binomial theorem from Taylor's.
2. To expand the logarithm of a binomial by Taylor's theorem.
3. To deduce the exponential theorem from Taylor's.

4. $a^{\sin. x} = 1 + la.\frac{x}{1} + la^2.\frac{x^2}{1.2} + (la^3 - la)\frac{x^3}{1.2.3} \ldots\ldots$

$+(la^4 - 4la^2)\frac{x^4}{1.2.3.4} + \&c.$

5. $a^{\overset{-1}{\sin. x}} = 1 + la.\frac{x}{1} + la^2.\frac{x^2}{1.2} + (la^3 + la)\frac{x^3}{1.2.3} \ldots\ldots$

$+ (la^4 + 4la^2)\frac{x^4}{1.2.3.4} + \&c.$

6. The coefficient of the $(n+1)^{\text{th}}$. term of the expansion of $\sec. x = \frac{1}{2}.\,A_{2n-2} - \frac{1}{1.2.3.4}.\,A_{2n-4} + \&c. \ldots\ldots\ldots$

$\pm \frac{1}{1.2\ldots.(2n-2)}.\,A_2 \mp \frac{1}{1.2.3\ldots.2n}.$

7. Let $y^3 - 3y + x = 0$, then

$$\left.\begin{array}{l} y = \frac{x}{3} + \frac{x^3}{3^4} + \frac{x^5}{3^6} + \&c. \\ \text{and } y = \pm\sqrt{3} - \frac{1}{6}x \mp \frac{\sqrt{3}}{9^2}x^2 \ldots\ldots \end{array}\right\}\text{are the algebraick values of } y.$$

$$\text{Also } y = -x^{\frac{1}{3}} - \frac{1}{x^{\frac{1}{3}}} - \frac{2}{1.2\,x} + \frac{4}{1.2.3x^{\frac{4}{3}}} \ldots\ldots$$

8. $y^3 + a^2y - 2a^3 + axy - x^3 = 0$,

$$\therefore y = a - \frac{x}{2^2} + \frac{x^2}{2^6a} - \frac{131x^3}{2^9a^2} + \&c.$$

$$\text{also } y = x + \frac{a}{3} + \frac{2a^2}{3^2x} + \&c.$$

9. $y^3 - a^2y + axy - x^3 = 0$,

$$\therefore y = x - \frac{a}{3} + \frac{a^2}{3x} + \frac{a^3}{81x^2} \quad \frac{8a^4}{243x^3} \ldots\ldots$$

$$\text{also } y = \begin{cases} \pm a \mp \frac{x}{2} \mp \frac{x^2}{8a} + \ldots\ldots \\ -\frac{x^3}{a^2} - \frac{x^4}{a^3} - \frac{x^5}{a^4} - \frac{x^6}{a^5} \ldots\ldots \end{cases}$$

10. $xy^2 - ay = bx^3 + cx^2 + ex + g$

$$\therefore y = -\frac{g}{a} + \frac{g^2 - a^2e}{a^3}x - \frac{1}{a^5}(2g(g^2 - a^2e) + a^4c)x^2 + \&c.$$

$$\text{and } = \frac{g}{ax} - \frac{g^2 - a^2e}{a^3} + \frac{1}{a^5}(2g(g^2 - a^2e) + a^4c)x + \&c.$$

Also $y = b^{\frac{1}{2}}x + \dfrac{c}{2b^{\frac{1}{2}}} + \dfrac{4ab^{\frac{3}{2}} + 4bc - e^2}{8b^{\frac{3}{2}}x} + \&c.$ which contains the *two* values that would result from the solution of the equation as a quadratick.

12. *Required to expand a function of two independent variables.*

Suppose the principal variables to change separately.

Let $u = \text{F}(x, y)$

$u_{\prime} = \text{F}(x + h, y)$

$\text{U} = \text{F}(x + h, y + k)$

By Taylor's Theor. Art. 5, we have,

$u_{\prime} = u + \dfrac{du}{dx}\dfrac{h}{1} + \dfrac{d^2u}{dx^2}\dfrac{h^2}{1.2} + \dfrac{d^3u}{dx^3}\dfrac{h^3}{1.2.3} + \&c.$, where the coefficients are functions of x and y independent of h.

Next, suppose y to become $y + k$, then each term of this equation will receive a change, $u_{\prime}$ becomes U, and all the remaining terms may be expanded by the same Theorem.

$$u \text{ becomes } u + \frac{du}{dy}\frac{k}{1} + \frac{d^2u}{dy^2}\frac{k^2}{1.2} + \frac{d^3u}{dy^3}\frac{k^3}{1.2.3} + \&c.$$

$$\frac{du}{dx} \quad \text{——} \quad \frac{du}{dx} + \frac{d.\frac{du}{dx}}{dy}.\frac{k}{1} + \frac{d^2.\frac{du}{dx}}{dy^2}.\frac{k^2}{1.2} + \&c.$$

$$\frac{d^2u}{dx^2} \quad \text{——} \quad \frac{d^2u}{dx^2} + \frac{d.\frac{d^2u}{dx^2}}{dy}.\frac{k}{1} + \&c.$$

$$\frac{d^3u}{dx^3} \quad \text{——} \quad \frac{d^3u}{dx^3} + \&c.$$

$$\&c. \quad \text{——} \quad \&c.$$

Hence the equation becomes

$$\text{U} = u + \frac{du}{dy}\cdot\frac{k}{1} + \frac{d^2u}{dy^2}\frac{k^2}{1.2} + \frac{d^3u}{dy^3}\frac{k^3}{1.2.3} + \&c.$$

$$+\frac{h}{1}\left\{\frac{du}{dx} + \frac{d.\frac{du}{dx}}{dy}\cdot\frac{k}{1} + \frac{d^2.\frac{du}{dx}}{dy^2}\cdot\frac{k^2}{1.2} + \&c.\right\}$$

$$+\frac{h^2}{1.2}\left\{\frac{d^2u}{dx^2} + \frac{d.\frac{d^2u}{dx^2}}{dy}\cdot\frac{k}{1} + \&c.\right\}$$

$$+\frac{h^3}{1.2.3}\left\{\frac{d^3u}{dx^3} + \&c.\right\}$$

$$+ \&c.$$

13. In the preceding expansion, all the fluxional coefficients contain partial fluxions of the function. Thus in the term $\frac{du}{dy}\frac{k}{1}$, du expresses the fluxion of u taken only with respect to y, for it is obtained from $u = \text{F}(x, y)$ on the supposition that y alone varies. So also in the term $\frac{du}{dx}\frac{h}{1}$, du does not express the whole fluxion of u, but that part which arises from supposing x to vary and y to be constant. In the term $\frac{d^2u}{dy^2}\frac{k^2}{1.2}$, d^2u is the second fluxion of the function taken in each case with respect to y.

There is also another point relating to notation which should be particularly observed. It is usual to denote $\frac{d.\frac{du}{dx}}{dy}$ by $\frac{d^2u}{dx\,dy}$ *, which therefore signifies that the d^2u in this case is obtained by first differentiating with respect to x, and then differentiating the result with respect to y. $\frac{d^2u}{dy\,dx}$, which is the same as $\frac{d.\frac{du}{dy}}{dx}$, would show that the order

* Lacroix and other writers denote $\frac{d.\frac{du}{dx}}{dy}$ by $\frac{d^2u}{dy\,dx}$.

of the operations is to be inverted. These two symbols *may* represent equal quantities, but they are not identical, since they indicate different operations.

$\dfrac{d^2 . \dfrac{du}{dx}}{dy^2} = \dfrac{d^3u}{dxdy^2}$, which shows that u is to be differentiated thrice; first with respect to x, and then twice with respect to y.

Generally $\dfrac{d^{m+n}u}{dx^p dy^n dx^q}$, where $p + q = m$, shows that u is to be differentiated $m+n$ times, the first p times with respect to x, then n times with respect to y, and then q times with respect to x.

The preceding series then may be expressed thus,

$$U = u + \left(\frac{du}{dx}\frac{h}{1} + \frac{du}{dy}k\right) + 1.2\left(\frac{d^2u}{dx^2}h^2 + \frac{d^2u}{dxdy}2hk + \frac{d^2u}{dy^2}k^2\right)$$
$$+ 1.2.3\left(\frac{d^3u}{dx^3}h^3 + \frac{d^3u}{dx^2dy}3h^2k + \frac{d^3u}{dxdy^2}3hk^2 + \frac{d^3u}{dy^3}k^3\right) + \&c.$$

which is the same series as that which was deduced, Ch. 3. 6., upon principles independent of the theory of limiting ratios.

The terminus generalis is $\dfrac{h^m k^n}{(1.2.3 \ldots . m)(1.2.3 \ldots . n)} \times \dfrac{d^{m+n}u}{d^m x\, d^n y}$.

14. In calculating with partial fluxions the student cannot be too cautious even in performing the simplest algebraical operations. Thus, if we have $\dfrac{dy}{dx} = \text{A}$, and consequently $1 = \dfrac{\text{A}}{\dfrac{dy}{dx}}$, we are not to conclude without proof that $1 = \text{A}\dfrac{dx}{dy}$; for here we shift the hypothesis. That the conclusion, however, would be just appears from the following demonstration.

Let y' and x' be corresponding values of y and x; then h being the increment of x, it appears from Ch. 3. 1. that we may assume $\dfrac{y'-y}{x'-x} = \text{A} + \text{B}h + \text{C}h^2 + \&c.$, where A, B, C, &c. are functions of y', y and x independent of h; taking

the reciprocals we have $\frac{x'-x}{y'-y} = \frac{1}{\text{A}+\text{B}h+\text{C}h^2+\&\text{c.}}$ =, by division, $\frac{1}{\text{A}} - \frac{\text{B}}{\text{A}^2}.h$ + &c.; hence, taking these ratios in their limit, we have $\frac{dx}{dy} = \frac{1}{\text{A}}$; but from the original assumption $\frac{dy}{dx} = \text{A}$, therefore $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

Or thus.

Let $u = \text{F}y$ and $y = fx$, then (3.8) $du = \frac{du}{dy}\frac{dy}{dx}dx$, or $\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}$.

Make $u = x$, and we have $1 = \frac{dx}{dy}.\frac{dy}{dx}$ or $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

15. *The total fluxion of a function of two independent variables is equal to the sum of its partial fluxions.*

For (Art. 12.) $\text{U} = u + \left(\frac{du}{dx}\frac{h}{1} + \frac{du}{dy}\frac{k}{1}\right)$ ....
$+1.2\left(\frac{d^2u}{dx^2}h^2 + \frac{d^2u}{dx\,dy}2hk + \frac{d^2u}{dy^2}k^2\right)$ + &c.; hence, transposing u and dividing by inc. u, and taking the ratios in their limit, there results $1 = \frac{du}{dx}\frac{dx}{du} + \frac{du}{dy}\frac{dy}{du}$ or ....
$du = \frac{du}{dx}dx + \frac{du}{dy}dy$.

Instances of this theorem have already been demonstrated in the first chapter: thus $d(x+y) = dx + dy$; $d.xy = ydx + xdy$; $d.\frac{x}{y} = \frac{dx}{y} - \frac{xdy}{y^2}$; $d.y^x = ly\,y^x dx + xy^{x-1}dy$.

Examples.

Ex. 1. $u = x^m y^n$.

$$\left.\begin{aligned}\frac{du}{dx}dx &= mx^{m-1}y^n dx\\ \&\frac{du}{dy}dy &= nx^m y^{n-1}dy\end{aligned}\right\} \therefore du = mx^{m-1}y^n dx + nx^m y^{n-1}dy.$$

Ex. 2. $u = \dfrac{ay}{\sqrt{x^2+y^2}}$.

$$\frac{du}{dx}dx = -\frac{ayxdx}{(x^2+y^2)^{\frac{3}{2}}}$$

$$\& \frac{du}{dy}dy = \frac{ady}{\sqrt{x^2+y^2}} - \frac{ay^2dy}{(x^2+y^2)^{\frac{3}{2}}} \quad \therefore du = \frac{-ayxdx + ax^2dy}{(x^2+y^2)^{\frac{3}{2}}}.$$

Ex. 3. $u = \tan.^{-1}\dfrac{x}{y}$.

$$\left.\begin{aligned} \frac{du}{dx}dx &= \frac{\frac{dx}{y}}{1+\frac{x^2}{y^2}} = \frac{ydx}{x^2+y^2} \\ \& \frac{du}{dy}dy &= \frac{-\frac{xdy}{y^2}}{1+\frac{x^2}{y^2}} = \frac{-xdy}{x^2+y^2} \end{aligned}\right\} \therefore du = \frac{ydx - xdy}{x^2+y^2}.$$

Ex. 4. $u = l.\tan.\dfrac{x}{y}$.

$$\frac{du}{dx}dx = \frac{d.\tan.\frac{x}{y}}{\tan.\frac{x}{y}} = \frac{\sec.^2\frac{x}{y}.\frac{dx}{y}}{\tan.\frac{x}{y}} = \frac{dx}{y\sin.\frac{x}{y}\cos.\frac{x}{y}}$$

$$\& \frac{du}{dy}dy = \frac{\sec.^2\frac{x}{y}.\frac{-xdy}{y^2}}{\tan.\frac{x}{y}} = \frac{-xdy}{y^2\sin.\frac{x}{y}\cos.\frac{x}{y}}$$

$$\therefore du = \frac{ydx - xdy}{y^2\sin.\frac{x}{y}\cos.\frac{x}{y}}.$$

16. *In taking partial fluxions of a function, the order in which they are taken will not affect the result.*

In Art. 12, we first changed x into $x+h$, and then y into $y+k$, in order to expand U; it is manifest that we shall

have the same value for U, if we invert the order of the substitutions, and in this case

$$
\begin{aligned}
\text{U} = u &+ \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3} + \&c. \\
&+ \frac{du}{dy}\frac{k}{1} + \frac{d^2u}{dydx}\frac{k}{1}\cdot\frac{h}{1} + \frac{d^3u}{dy\,dx^2}\frac{k}{1}\cdot\frac{h^2}{1.2} + \&c. \\
&+ \frac{d^2u}{dy^2}\cdot\frac{k^2}{1.2} + \frac{d^3u}{dy^2dx}\frac{k^2}{1.2}\frac{h}{1} + \&c. \\
&+ \frac{d^3u}{dy^3}\cdot h^3 + \&c. \\
&+ \&c.
\end{aligned}
$$

This series being identical with the one in Art. 12, we have, equating the coefficients of corresponding terms, $\frac{d^2u}{dydx} = \frac{d^2u}{dx\,dy}$, $\frac{d^3u}{dy\,dx^2} = \frac{d^3u}{dx^2\,dy}$, and generally $\frac{d^{m+n}u}{dx^m dy^n}$ $= \frac{d^{m+n}u}{dy^n dx^m}$.

We also have $\frac{d^3u}{dy\,dx^2} = \frac{d^3u}{dydxdx} = \frac{d.\frac{d^2u}{dydx}}{dx} = \frac{d.\frac{d^2u}{dx\,dy}}{dx}$ $= \frac{d^3u}{dx\,dy\,dx}$, which shows that the result is not affected by the order of the differentiations.

It is evident that the same conclusion obtains whatever be the nature of x and y, whether independent or not.

Examples.

1. To show that $\frac{d^2u}{dxdy} = \frac{d^2u}{dydx}$.

(1.) Let $u = x^m y^n$, then $\frac{du}{dx} = my^n x^{m-1}$ $\therefore \frac{d^2u}{dxdy} = mnx^{m-1}y^{n-1}$;

also $\frac{du}{dy} = nx^m y^{n-1}$ $\therefore \frac{d^2u}{dydx} = mnx^{m-1}y^{n-1} = \frac{d^2u}{dxdy}$.

(2.) $u = \sqrt{1+ax}\,\sqrt{1+by}$ $\therefore \frac{d^2u}{dxdy} = \frac{ab}{4\sqrt{1+ax}\,\sqrt{1+by}}$ $= \frac{d^2u}{dydx}$.

(3.) $u = x \sin.y + y \sin.x \therefore \frac{d^2u}{dx\,dy} = \cos.x + \cos.y = \frac{d^2u}{dydx}$.

(4.) $u = x\sqrt{2xy + y^2} \therefore \frac{d^2u}{dxdy} = \frac{d^2u}{dydx} = \frac{3x^2y + 3xy^2 + y^3}{(2xy + y^2)^{\frac{3}{2}}}$;

and $\frac{d^3u}{dx^2dy} = \frac{3x^2y^2 + 3xy^3}{(2xy + y^2)^{\frac{5}{2}}} = \frac{d^3u}{dx\,dy\,dx}$.

17. *Required to investigate the conditions necessary in order that two given functions of the variables may be considered as partial coefficients of a third function of the same variables.*

Let $F(x, y)$, $f(x, y)$ represent the two given functions; $\varphi(x, y)$ a third function of the same variables; and, adopting Lagrange's notation, in order that we may make the two suppositions $F(x, y) = \varphi'(x, y)$, and $f(x, y) = \varphi_{\prime}(x, y)$, it is evident that we must have $F_{\prime}(x, y)$, which $= \varphi_{\prime}'(x, y)$, equal to $f'(x, y)$.

Generally, in order that we may suppose $F(x, y) = \varphi_n^m(x, y)$ and $f(x, y) = \varphi_q^p(x, y)$, a necessary condition is, that $F_q^p(x, y)$, which $= \varphi_{n+q}^{m+p}(x, y)$, is equal to $f_n^m(x, y)$.

Ex. Let $F(x, y) = \frac{y}{x^2 + y^2}$ and $f(x, y) = \frac{-x}{x^2 + y^2}$, ...

$\therefore F_{\prime}(x, y) = \frac{1}{x^2 + y^2} - \frac{2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$, ...

and $f'(x, y) = -\frac{1}{x^2 + y^2} + \frac{2x^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$; hence we may make the suppositions that $F(x, y) = \varphi'(x, y)$ and $f(x, y) = \varphi_{\prime}(x, y)$, φ representing any other function whatever of x and y: but we may not suppose $F(x, y) = \varphi_{\prime}'(x, y)$ and $f(x, y) = \varphi''(x, y)$, since $F'(x, y)$ does not equal $f_{\prime}(x, y)$.

18. *Required to find the successive fluxions of a function of two variables, on the supposition that they both flow uniformly.*

$du = \frac{du}{dx}dx + \frac{du}{dy}dy$; therefore

$d^2u = d.\frac{du}{dx} \times dx + d.\frac{du}{dy} \times dy$;

but $d.\frac{du}{dx} =$ (Art. 15) $\frac{d^2u}{dx^2}dx + \frac{d^2u}{dxdy}dy,$

and $d.\frac{du}{dy} = \frac{d^2u}{dydx}dx + \frac{d^2u}{dy^2}dy;$

hence $d^2u = \frac{d^2u}{dx^2}dx^2 + \frac{2d^2u}{dxdy}dxdy + \frac{d^2u}{dy^2}dy^2.$

Similarly d^3u may be proved $= \frac{d^3u}{dx^3}dx^3 + \frac{3d^3u}{dx^2dy}dx^2dy + \frac{3d^3u}{dxdy^2}dxdy^2 + \frac{d^3u}{dy^3}dy^3$, where the law of the terms is sufficiently manifest, and the rule for the expansion of d^nu may be demonstrated by the method of induction.

Cor. If in the expansion of $\mathrm{U} = \mathrm{F}(x+h, y+k)$, Art. 12, we substitute dx for h, and dy for k, we have

$$\mathrm{U} = u + \frac{1}{1}\left\{\frac{du}{dx}dx + \frac{du}{dy}dy\right\} + \frac{1}{1.2}\left\{\frac{d^2u}{dx^2}dx^2 + \frac{d^2u}{dxdy}2dxdy + \frac{d^2u}{dy^2}dy^2\right\} + \frac{1}{1.2.3}\left\{\frac{d^3u}{dx^3}dx^3 + \frac{d^3u}{dx^2dy}3dx^2dy + \frac{d^3u}{dxdy^2}3dxdy^2 + \frac{d^3u}{dy^3}dy^3\right\} + \&c.;$$

and substituting for these terms their value obtained in the article, we have $\mathrm{U} = u + \frac{du}{1} + \frac{d^2u}{1.2} + \frac{d^3u}{1.2.3} + \&c.$, the same series as that deduced Art. 5. Cor. 2.

19. $$f(x, y) = f. + (xf.' + yf._{\prime}) + \frac{1}{1.2}(x^2f.'' + 2xyf._{\prime}' + y^2f._{\prime\prime}) + \frac{1}{1.2.3}(x^3f.''' + 3x^2yf._{\prime}'' + 3xy^2f._{\prime}'' + y^3f._{\prime\prime}') + \&c.$$

where $f.$, $f.'$, $f._{\prime}$, &c. represent the values of the partial coefficients of $u = f(x, y)$ when $x = 0$ and $y = 0$.

For the general form of the series which is the expansion of u is

$$\left.\begin{aligned} u = \mathrm{A} &+ \mathrm{B}x + \mathrm{C}y \\ &+ \mathrm{D}x^2 + \mathrm{E}xy + \mathrm{F}y^2 \\ &+ \&c. \end{aligned}\right\}$$ where the coefficients A, B, C, &c. are independent of x and y.

To determine these, first suppose $x = 0$ and $y = 0$, then $\mathrm{A} = (u) = f.$; next, differentiate partially with respect to x,

and there results $\frac{du}{dx} = \text{B} + 2\text{D}x + \text{E}y + \&c.$; in which, if we suppose $x = 0$ and $y = 0$, there results $\text{B} = f.'$; and in the same manner the remaining coefficients may be deduced.

Or the Theorem may be derived from Art. 12, as Maclaurin's Theorem is deduced from Taylor's in Art. 7.

20. *To develope a function of three independent variables.*

Let $u = f(x, y, z)$ and $\text{U} = f(x+h, y+k, z+l)$, and suppose the variables to change separately; then it may be shown, as in Art. 12, that

$$f(x+h, y+k, z)$$
$$= u + \left(\frac{du}{dx}h + \frac{du}{dy}k\right) + \frac{1}{1.2}\left(\frac{d^2u}{dx^2}h^2 + \frac{d^2u}{dxdy}2hk + \frac{d^2u}{dy^2}k^2\right)$$
$$+ \frac{1}{1.2.3}\left(\frac{d^3u}{dx^3}h^3 + \frac{d^3u}{dx^2dy}3h^2k + \frac{d^3u}{dxdy^2}3hk^2 + \frac{d^3u}{dy^3}k^3\right) + \&c.$$

Next, let z become $z + l$, then

$$u \text{ becomes } u + \frac{du}{dz}\frac{l}{1} + \frac{d^2u}{dz^2}\frac{l^2}{1.2} + \frac{d^3u}{dz^3}\frac{l^3}{1.2.3} + \&c.$$

$$\frac{du}{dx} \text{ becomes } \frac{du}{dx} + \frac{d^2u}{dxdz}\cdot\frac{l}{1} + \frac{d^3u}{dxdz^2}\frac{l^2}{1.2} + \&c.$$

&c. — &c.

Hence

$$\text{U} = u + \frac{du}{dz}\frac{l}{1} + \frac{d^2u}{dz^2}\frac{l^2}{1.2} + \frac{d^3u}{dz^3}\frac{l^3}{1.2.3} + \&c.$$
$$+ \frac{h}{1}\cdot\left\{\frac{du}{dx} + \frac{d^2u}{dxdz}\frac{l}{1} + \frac{d^3u}{dxdz^2}\frac{l^2}{1.2}\right\}$$
$$+ \frac{k}{1}\left\{\frac{du}{dy} + \frac{d^2u}{dydz}\frac{l}{1} + \frac{d^3u}{dydz^2}\frac{l^2}{1.2}\right\}$$
$$+ \frac{h^2}{1.2}\left\{\frac{d^2u}{dx^2} + \frac{d^3u}{dx^2dz}\frac{l}{1}\right\}$$
$$+ hk\left\{\frac{d^2u}{dxdy} + \frac{d^3u}{dxdydz}\frac{l}{1}\right\}$$
$$+ \frac{k^2}{1.2}\left\{\frac{d^2u}{dy^2} + \frac{d^3u}{dy^2dz}\frac{l}{1}\right\}$$
$$+ \frac{1}{1.2.3}\left\{\frac{d^3u}{dx^3}h^3 + \frac{d^3u}{dx^2dy}3h^2k + \frac{d^3u}{dxdy^2}3hk^2 + \frac{d^3u}{dy^3}k^3\right\}$$
$$+ \&c.$$

neglecting those terms which involve more than three dimensions of h, k, and l. Hence, arranging the terms according to the dimensions of h, k, and l, we have

$$\text{U}=u+\frac{1}{1}\left\{\frac{du}{dx}h+\frac{du}{dy}k+\frac{du}{dz}l\right\}+\frac{1}{1.2}\left\{\frac{d^2u}{dx^2}h^2+\frac{d^2u}{dy^2}k^2\right.$$

$$+\frac{d^2u}{dz^2}l^2+\frac{d^2u}{dxdy}2hk+\frac{d^2u}{dxdz}2hl+\frac{d^2u}{dydz}2kl\Big\}+\frac{1}{1.2.3}\left\{\frac{d^3u}{dx^3}h^3\right.$$

$$+\frac{d^3u}{dy^3}k^3+\frac{d^3u}{dz^3}l^3+\frac{d^3u}{dx^2dy}3h^2k+\frac{d^3u}{dx^2dz}3h^2l+\frac{d^3u}{dy^2dz}3k^2l$$

$$+\frac{d^3u}{dxdy^2}3hk^2+\frac{d^3u}{dxdz^2}3hl^2+\frac{d^3u}{dydz^2}3kl^2+\frac{d^3u}{dxdydz}\cdots$$

$$\left.\times 2.3hkl.\right\}+\&c.$$

In the application of the calculus, it is seldom required to develope a function of more than two variables; for, as the place of a point in fixed space is, in general, made to depend upon the value of three rectangular co-ordinates, the equations are of the form $\text{F}(x, y, z) = 0$, from which we may obtain $z = f(x, y)$, which may be developed as in Art. 12.

21. *The total fluxion of a function of three or more independent variables is equal to the sum of its partial fluxions.*

For in the above series transpose u, divide by inc. u, and take the ratios in their limit, and there will result

$$1=\frac{du}{dx}\frac{dx}{du}+\frac{du}{dy}\frac{dy}{du}+\frac{du}{dz}\frac{dz}{du},$$

$$\text{or } du=\frac{du}{dx}dx+\frac{du}{dy}dy+\frac{du}{dz}dz.$$

It is evident that this demonstration may be extended to a function of any number of variables.

22. Hitherto we have considered u as a function of two or more *independent* variables x, y, z, &c. If the variables are dependent upon each other in any manner, the partial fluxions of u may appear under a different form, according to the hypotheses which may be made with respect to the variables; but by means of the formulæ of Articles 12 and 20, the truth of the proposition may in all cases be established, that the total fluxion is equal to the sum of its partial fluxions.

Ex. 1. Let $u = \text{F}(x, y)$, and suppose $y = fx$, then, Art. 12,

$$\text{U} = u + \left(\frac{du}{dx}\frac{h}{1} + \frac{du}{dy}\frac{k}{1}\right) + 1.2\left(\frac{d^2u}{dx^2}h^2 + \frac{d^2u}{dxdy}2hk + \frac{d^2u}{dy^2}k^2\right)$$

&c.; transpose u, divide by h, and take the ratios in their limit, there results $\dfrac{\dot{d}u}{dx} = \dfrac{du}{dx} + \dfrac{du}{dy}\dfrac{dy}{dx}$

$$\text{and } \dot{d}u = \frac{du}{dx}dx + \frac{du}{dy}\frac{dy}{dx}dx,$$

$$\text{or } = \frac{du}{dx}dx + \frac{du}{dy}dy.$$

Ex. 2. Let $u = \text{F}(x, y, z)$, and let y & z be each functions of x.

$$(\text{Art. } 20.)\ \text{U} = u + \left(\frac{du}{dx}\frac{h}{1} + \frac{du}{dy}\frac{k}{1} + \frac{du}{dz}\frac{l}{1}\right) + \text{\&c.};$$

therefore, as in Ex. 1. $\dfrac{\dot{d}u}{dx} = \dfrac{du}{dx} + \dfrac{du}{dy}\dfrac{dy}{dx} + \dfrac{du}{dz}\dfrac{dz}{dx}$

$$\text{and } \dot{d}u = \frac{du}{dx}dx + \frac{du}{dy}\frac{dy}{dx}dx + \frac{du}{dz}\frac{dz}{dx}dx;$$

$$\text{or } = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz.$$

Ex. 3. Let $u = \text{F}(x, y, z)$ and $z = f(x, y)$.

Transpose u and divide by l, and there results

$$\frac{\dot{d}u}{\dot{d}z} = \frac{du}{\dot{d}z} + \frac{du}{dx}\frac{dx}{\dot{d}z} + \frac{du}{dy}\frac{dy}{\dot{d}z}$$

$$\text{or } \dot{d}u = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz.$$

And it is obvious that whatever be the number of variables, or whatever hypothesis may be made with respect to them, the proposition may be demonstrated.

23. *If* u *is a function of two dependent variables* x *and* y, *then shall* $\text{U} = u + \dfrac{\dot{d}u}{dx}\dfrac{h}{1} + \dfrac{\dot{d}^2u}{dx^2}\dfrac{h^2}{1.2} + \dfrac{\dot{d}^3u}{dx^3}\dfrac{h^3}{1.2.3} + \text{\&c.}$

For, since $u=\text{F}(x, y)$ and $y = fx$, therefore $u = \text{F}(x, fx) = \varphi x$, and consequently by Taylor's theorem,

$$\text{U} = u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \&c.$$

Since the increment of u, and consequently its fluxion, must be the same, whether we suppose it to arise from the changes of x and y in $u = \text{F}(x, y)$, or from the change of x alone in $u = \varphi x$; it follows that the coefficients of h in this developement contain the *total* fluxions of u.

24. *Required to develope a function of two dependent variables in terms of the increment of one of them, so that the coefficients may contain partial fluxions of the function.*

Since $u = \text{F}(x, y)$ therefore, Art. 12,

$$\text{U}=u+\left\{\frac{du}{dx}h+\frac{du}{dy}k\right\}+\frac{1}{1.2}\left\{\frac{d^2u}{dx^2}h^2+\frac{d^2u}{dxdy}2hk+\frac{d^2u}{dy^2}k^2\right\}$$
$$+\frac{1}{1.2.3}\left\{\frac{d^3u}{dx^3}h^3+\frac{d^3u}{dx^2dy}3h^2k+\frac{d^3u}{dxdy^2}3hk^2+\frac{d^3u}{dy^3}k^3\right\}+\&c.;$$

but since $y = fx$, therefore (Taylor's Theorem)

$$k=ph+qh^2+rh^3+\&c. \text{ where } p=\frac{dy}{dx},\ q=\frac{d^2y}{dx^2},\ r=\frac{d^3y}{dx^3},$$

&c. = &c.; which substitute in the preceding developement, and arranging the terms according to the powers of h, we have

$$\text{U}=u+\left\{\frac{du}{dx}+\frac{du}{dy}p\right\}.\frac{h}{1}+\left\{\frac{d^2u}{dx^2}+\frac{2d^2u}{dxdy}p+\frac{d^2u}{dy^2}p^2+\frac{du}{dy}q\right\}$$
$$\frac{h^2}{1.2}+\left\{\frac{d^3u}{dx^3}+\frac{3d^3u}{dx^2dy}p+\frac{3d^3u}{dxdy^2}p^2+\frac{d^3u}{dy^3}p^3+\frac{3d^2u}{dxdy}q\right.$$
$$\left.+\frac{3d^2u}{dy^2}pq+\frac{du}{dy}r\right\}\frac{h^3}{1.2.3}+\&c.$$

By the former article each coefficient should be the total fluxion of that which precedes it, divided by dx.

Now the second coefficient is the sum of the partial coefficients of u; and from this to deduce the third coefficient, we have the fluxion of $\frac{du}{dx}$, taken with respect to x and divided by dx, equal to $\frac{d^2u}{dx^2}$; and to find its fluxion taken

with respect to y, we have $\frac{du}{dx}$, a function of x and y and $y = fx$, and consequently (3. 8) the required fluxion is $\frac{d^2u}{dxdy}\frac{dy}{dx}dx$, which, divided by dx, is $\frac{d^2u}{dxdy}p$.

Again, to differentiate $\frac{du}{dy}p$, we have the partial fluxion of $\frac{du}{dy}$ taken with respect to x and divided by $dx = \frac{d^2u}{dydx}$; also, since $\frac{du}{dy}$ is a function of x and y and $y = fx$, its partial fluxion taken with respect to $y = \frac{d^2u}{dy^2}\frac{dy}{dx}dx$, which, divided by dx, is $\frac{d^2u}{dy^2}p$; hence $\frac{d^2u}{dx^2} = \frac{d^2u}{dx^2} + \frac{d^2u}{dxdy}p$
$+ p\left\{\frac{d^2u}{dydx} + \frac{d^2u}{dy^2}p\right\} + \frac{du}{dy}\frac{dp}{dx} = \frac{d^2u}{dx^2} + \frac{2d^2u}{dxdy}p + \frac{d^2u}{dy^2}p^2$
$+ \frac{du}{dy}q$.

And in the same manner the succeeding coefficients may be deduced.

This developement then, combined with that of the preceding article, shows that even when u is a function of *dependent* variables, its successive total fluxions are found by taking the sum of all the partial fluxions of each preceding term.

Similarly it may be shown that " *If* u *is a function of three or more variables dependent upon each other in any manner whatever, it may be developed in a series ascending by the powers of the increment of one of the variables, the form of whose coefficients will depend upon the assumed relation of the variables ; and each coefficient, which* (Art. 23) *is the total fluxion of that which precedes it divided by the fluxion of the principal variable, is also the sum of all its partial fluxions divided by the fluxion of the principal variable.*

Combining this with the proposition contained in Art. 21, it follows that, " u *being a function of any number of va-*

riables, whatever be their nature or relation, the total fluxion of d·u *is equal to the sum of its partial fluxions.*

This general theorem, which has been demonstrated in so many different instances, on account of its importance, and for the purposes of illustration, is an immediate inference from the definition of a fluxion given in Ch. 1. Art. 7.

For, since on every supposition that can be made with respect to the variables, the total increment of a function must equal the sum of all its partial increments, and that this must obtain whatever be the state of the increments, it follows from the definition that its total fluxion is the sum of its partial fluxions.

25. When a function is developed, as in the preceding article, the variable in terms of whose increment the series ascends, is called *the principal independent.*

It appears from the developement that the coefficients of the series contain different partial fluxions of u, and also different combinations of the fluxions of the other variables considered as functions of the principal independent; the former, in any proposed function, may always be had in terms of the variables free from fluxional coefficients; hence the coefficients of the series, or the total fluxions of u, divided by the fluxion of the principal independent, may always be expressed in terms of the variables, and of the fluxions of the variables, considered as functions of the principal independent.

Thus, suppose $u = x^3 + 3axy + y^3$;

then, since $\dfrac{du}{dx} = 3(x^2 + ay), \dfrac{du}{dy} = 3(ax + y^2), \dfrac{d^2u}{dx^2} = 2.3x,$

$\dfrac{d^2u}{dxdy} = 3a, \dfrac{d^2u}{dy^2} = 2.3y$, &c. = &c.;

therefore $\text{U} = u + \{3x^2 + 3ay + 3(ax + y^2)p\}\dfrac{h}{1} +$ &c.

from which values may be obtained for $\dfrac{\dot{d}u}{dx}, \dfrac{\dot{d}^2u}{dx^2}, \ldots\ldots$ in terms of $x, y, p, q, r, pq, \ldots\ldots$

26. *Fluxional Equations.*

Functions of dependent quantities usually appear under the form of equations $u = 0$, $v = 0$, ... which, with the conditions of the question, show the relation which the variables bear to each other; thus, from the equation

$y^2 - 2mxy + x^2 - a^2 = 0$ we may find the form of the function which y is of x. No other condition is necessary in this case for determining its form, since there are only two variables.

If there are three variables, we must have two equations in order to find the form of the function, which one is of either of the other two; and so on. (Alg. 145).

27. *If $u = 0$ represents a function of dependent variables, then shall $du = 0$, $d^2u = 0$, $d^3u = 0$, &c. &c.*

For since $u = \text{F}(x, y, z \ldots) = 0$, whatever be the values of $x, y, z \ldots$, therefore U, which $= \text{F}(x+h, y+k, z+l, \ldots)$ where $k, l, \ldots$ may be either positive or negative, $= 0$; and $\frac{\text{U}-u}{h}$, and consequently $\frac{du}{dx} = 0$, or $du = 0$: for the same reason, since $\frac{du}{dx} = f(x, y, z, \ldots)$, $d^2u = 0$, and so on.

Or thus. Since $\text{U} = 0$, therefore, (Art. 23)

$$u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \&c. = 0, \text{ \& consequently (Alg. 347)}$$

$du = 0$, $d^2u = 0$, $d^3u = 0$, &c. = &c.

Hence, if in the expansion in Art. 24 we suppose that $u = 0$, the coefficients of the series become the fluxional equations which arise from successively differentiating $u = 0$, on the supposition that x is the principal independent, and dividing each time by dx.

Since these coefficients contain partial fluxions of u, they are expressed in terms of the variables, and of different combinations of $p, q, r, \ldots$. It is under this or a similar form that fluxional equations always make their appearance in calculation; and the problem to be solved, when there is only one principal independent, is "Given the relation between the fluxional coefficients of the variables considered as functions of the principal independent; to find the primitive equation or the relation between the variables themselves."

Cor. All fluxional equations derived by successively differentiating $u = \text{F}(x, y) = 0$, on the supposition that x is the principal variable, and dividing each time by dx, are homogeneous with respect to the fluxions of y and the powers of dx. If y is taken to be the principal variable,

the equations, divided each time by dy, are homogeneous with respect to the fluxions of x and the powers of dy.

This appears from the developement in Art. 24.

It also follows from this view of the subject that we shall obtain the same values for $p, q, r, \ldots$, whether we differentiate $u = 0$, on the supposition that u is a function of dependent or of independent variables: for on either supposition, the sum of the partial fluxions is the same. But as this demonstration may perplex the student, in consequence of the sum of the partial fluxions being in each case equal to nothing, we shall make the proposition the subject of the next article.

28. *If we have an implicit function of* x *and* y, *viz.* $u = F(x, y) = 0$, *and consequently* $y = fx$; *in order to find the fluxional coefficients of* y *considered as a function of* x *from the equation* $u = 0$, *we may differentiate* u *as if* x *and* y *are independent variables, and from the resulting equations deduce the values of* $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, *&c.*

For (Art. 24.) $\dfrac{du}{dx} = \dfrac{du}{dx} + \dfrac{du}{dy}\dfrac{dy}{dx}$ or $du = \dfrac{du}{dx}dx + \dfrac{du}{dy}\dfrac{dy}{dx}dx$.

But since $y = fx$, $\dfrac{dy}{dx}dx = dy$; therefore we have

du, which $= 0$, $= \dfrac{du}{dx}dx + \dfrac{du}{dy}dy$ which is the value we should obtain for du in differentiating the function on the supposition that x and y are independent variables.

Also, since $\dfrac{du}{dx}$ and $\dfrac{du}{dy}$ are implicit functions of x and y and $y = fx$, it follows, as before, that we shall obtain the same value for d^2u, whether we differentiate du on the supposition that x and y are implicit or explicit functions; and consequently the resulting value of $\dfrac{d^2y}{dx^2}$ will be the same on either supposition. And the same may be proved true of $\dfrac{d^3y}{dx^3}, \dfrac{d^4y}{dx^4}$, &c. &c.

In these successive differentiations, $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}, \ldots$ or

their equals, $p, q, r, \ldots$ are considered as sole functions of x, so that $dp = qdx$, $dq = rdx$, &c. = &c.

The values obtained for $\dot{d}u$ on the two suppositions will not appear under the same *form*; because the symbols $\frac{du}{dy}dy$, and $\frac{du}{dy}\frac{dy}{dx}dx$, though algebraically equal, are not identical, since they do not indicate the same operation.

29. The advantage we derive from this mode of differentiating an equation is this; it enables us to calculate the values of $p, q, r, \ldots,$ without first solving the equation with respect to y.

To illustrate this, suppose $u = y^2 - 2mxy + x^2 - a^2 = 0$ or $y = mx \pm \sqrt{a^2 - x^2 + m^2x^2}$; then

$$\left.\begin{array}{l}\frac{du}{dx}dx = (-2my + 2x)dx, \text{ and} \\ \frac{du}{dy}dy = (2y - 2mx)dy\end{array}\right\}\text{therefore, dividing by 2, we}$$

have $du = 0 = (y - mx)dy - (my - x)dx$ (α), and $p = \frac{my - x}{y - mx}$, from which, if we eliminate y, there results . . . .

$$p = \frac{-x + m^2x \pm m\sqrt{a^2 - x^2 + m^2x^2}}{\pm\sqrt{a^2 - x^2 + m^2x^2}} = m \pm \frac{-x + m^2x}{\sqrt{a^2 - x^2 + m^2x^2}},$$

which are the values we should have obtained for p, had we differentiated on the supposition that $y = fx$.

Dividing (α) by dx, it becomes $\frac{\dot{d}u}{dx} = (y - mx)p - (my - x)$, which, differentiated on the supposition that x and y are independent quantities, and that p is a function of x, gives, dividing by dx, $\frac{\dot{d}^2u}{dx^2} = 0 = (y - mx)q + p(p - m) - mp + 1$, or $(y - mx)q + p^2 - 2mp + 1 = 0$, in which, if we substitute for y and for p their values in terms of x, we shall have the two values of q in terms of x; and they will prove to be the same as if we had obtained q by differentiating $y = fx$ twice successively: and so on.

In this example, the equation is so readily solved that

it would have been a shorter process to have found y in terms of x; and then, by successive differentiations, to have calculated $p, q, r, \ldots$; we shall therefore take a more complicated example, $x^3 + 3axy + y^3 = 0$; and suppose that it were required to find $p, q, r, \ldots$, without knowing the form of the function that y is of x.

$du = 0 = 3x^2dx + 3aydx + 3axdy + 3y^2dy$; therefore, omitting the 3, (α) $\dfrac{du}{dx} = 0 = x^2 + ay + (ax + y^2)p$, or

$p = -\dfrac{x^2 + ay}{ax + y^2}$; from which we cannot eliminate y, unless we know the form of the function that y is of x.

Differentiating (α) and dividing by dx, we have

$$\begin{aligned} \frac{d^2u}{dx^2} = 0 &= 2x + ap + (ax + y^2)q + p(a + 2yp) \\ &= 2x + 2ap + 2yp^2 + (ax + y^2)q; \end{aligned}$$

in which substitute $p = -\dfrac{x^2 + ay}{ax + y^2}$, and we have

$$\begin{aligned} 0 &= 2x - 2a.\frac{x^2 + ay}{ax + y^2} + 2y.\frac{(x^2 + ay)^2}{(ax + y^2)^2} + (ax + y^2)q \\ &= 2xy^4 + 6ax^2y^2 + 2x^4y - 2a^3xy + (ax + y^2)^3q \\ &= 2xy(y^3 + 3axy + x^3) - 2a^3xy + (ax + y^2)^3q \\ &= -2a^3xy + (ax + y^2)^3q \quad \therefore q = \frac{2a^3xy}{(ax + y^2)^3}: \text{ \& so on.} \end{aligned}$$

30. It is obvious that the same principle may be applied to a function of any number of variables $F(x, y, z \ldots t) = 0$, all of which are dependent upon one of them as t. It may be differentiated as if the variables were independent quantities, and the resulting equation divided by dt will give the relation between the first fluxional coefficients of the variables considered as functions of t. Also, since $\dfrac{dx}{dt}, \dfrac{dy}{dt}, \ldots$ are functions of $x, y, z \ldots t$, the equation may be again differentiated on the same supposition, and, dividing by dt, there will result an equation $\dfrac{d^2u}{dt^2} = 0$, which expresses the relation between the second fluxional coefficients of the variables considered as functions of t: and so on.

31. *Given a function of three variables, required to find the first fluxional coefficients of one of the variables considered as a function of the other two.*

Let the function be $u = \mathrm{F}(x, y, z) = 0$, and $z = f(x, y)$.

In Art. 20 suppose y constant, then we have

$$0 = \frac{du}{dx} + \frac{du}{dz}\,\frac{l}{h} = \frac{du}{dx} + \frac{du}{dz}\,\frac{dz}{dx}, \text{ or } \frac{dz}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dz}}.$$

Next, suppose x to be constant, then

$$0 = \frac{du}{dy} + \frac{du}{dz}\,\frac{dz}{dy} \text{ or } \frac{dz}{dy} = -\frac{\frac{du}{dy}}{\frac{du}{dz}}.$$

All the fluxional coefficients contained in these equations are partial.

Cor. Multiplying the first equation by dx, and the second by dy, and adding the results, we have

$$0 = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}\left\{\frac{dz}{dx}dx + \frac{dz}{dy}dy\right\};$$

but $\frac{dz}{dx}dx + \frac{dz}{dy}dy = dz$, therefore we have

$0 = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz$, which corroborates what has been already demonstrated, Art. 29.

32. It is obvious that x and y, which are here considered as the principal variables, must be independent of each other: they must be so related that the one can be supposed to change without necessarily producing a change in the other.

Thus, if we have only the function $x^2 + y^2 + z^2 - r^2 = 0 = u$, y may be supposed to vary, and the conditions of the equation may be satisfied by assigning a corresponding change to z, x remaining constant; but if we have another equation between the variables, for instance $x^2 + y^2 = 2az$, by eliminating z, we shall have an equation of the form $f(x, y) = 0$, in which the one is necessarily a *determinate* function of the other, and the hypotheses made in the preceding article are inadmissible, and the fluxional equations do not obtain.

33. *Lagrange's method of deducing the fluxional equations of Art. 30.*

Since $u = \text{F}(x, y, z) = 0$, therefore, Art. 29,

$$du = 0 = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz\,;$$

but since $z = f(x, y)$, $dz = \dfrac{dz}{dx}dx + \dfrac{dz}{dy}dy$, therefore

$$du = 0 = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}\left\{\frac{dz}{dx}dx + \frac{dz}{dy}dy\right\}$$

$$= \left\{\frac{du}{dx} + \frac{du}{dz}\frac{dz}{dx}\right\}dx + \left\{\frac{du}{dy} + \frac{du}{dz}\frac{dz}{dy}\right\}dy$$

in which equation dx and dy are considered as independent quantities, consequently (Alg. 346)

$$\frac{\ddot{d}u}{dx} = 0 = \frac{du}{dx} + \frac{du}{dz}\frac{dz}{dx}\ (\alpha)$$

$$\text{and}\quad \frac{\ddot{d}u}{dy} = 0 = \frac{du}{dy} + \frac{du}{dz}\frac{dz}{dy}.\ (\beta)$$

(Fonct. Anal. p. 141.)

Hence it appears that from an equation containing three variables, there may be derived *two* independent fluxional equations of the first order, which are simultaneous, and that their sum is the same fluxional equation, though under a different form, as that which would arise from differentiating the function on the supposition that all the variables are independent.

34. Since in the equations (α) and (β) all the partial fluxional coefficients contain new functions of x, y, z, they may be again differentiated as in the preceding article, on the two hypotheses that x is constant and that y is constant. And it may be here observed, that though (α) has been derived on the supposition that y is constant, yet as it is of the form $\text{F}\left(x, y, z, \dfrac{dz}{dx}\right) = 0$, or $\text{F}(x, y, z, s) = 0$, where $s = f(x, y, z)$, an equation which obtains for all values of the variables, and in which we may make any hypothesis with respect to them which is not inconsistent with the conditions of the question, it follows that it may be differentiated as if x were constant and y the principal variable.

If we differentiate (α) and (β) on these two hypotheses,

there will result four equations: two of them will prove to be identical, since $\frac{\ddot{d}^2u}{dx\,dy} = \frac{\ddot{d}^2u}{dy\,dx}$; so that only three fluxional equations of the second order can be deduced *immediately* from $u = 0$.

These equations express the relation between $\frac{dz}{dx}$, $\frac{dz}{dy}$, $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dx\,dy}$ . . . ., which are to be considered as new functions of x, y, z; and treating them as such, and proceeding as before, there will arise a succession of fluxional equations, showing the relation between the fluxional coefficients of z, considered as a function of two independent variables x and y.

These equations are to be derived as in Art. 24, by considering each partial coefficient as a function of z, x, or of z, y and z, as a function of x or y, according as y or x is supposed to be constant.

The fluxional equations of the second order are

$$* \frac{\ddot{d}^2u}{dx^2} = 0 = \frac{d^2u}{dx^2} + 2\frac{d^2u}{dx\,dz}\frac{dz}{dx} + \frac{d^2u}{dz^2}\frac{dz^2}{dx^2} + \frac{du}{dz}\frac{d^2z}{dx^2}.$$

$$\frac{\ddot{d}^2u}{dx\,dy} = 0 = \frac{d^2u}{dx\,dy} + \frac{d^2u}{dz\,dy}\frac{dz}{dx} + \frac{d^2u}{dz\,dx}\frac{dz}{dy} + \frac{d^2u}{dz^2}\frac{dz}{dx}\frac{dz}{dy} + \frac{du}{dz}\frac{d^2z}{dx\,dy}.$$

$$\frac{\ddot{d}^2u}{dy^2} = 0 = \frac{d^2u}{dy^2} + 2\frac{d^2u}{dy\,dz}\frac{dz}{dy} + \frac{d^2u}{dz^2}\frac{dz^2}{dy^2} + \frac{du}{dz}\frac{d^2z}{dy^2}.$$

Cor. If these equations be multiplied respectively by dx^2, $2dx\,dy$, and dy^2, and the results be added together, they will form an equation, in which, if dz be substituted for $\frac{dz}{dx}dx + \frac{dz}{dy}dy$, and d^2z, for $\frac{d^2z}{dx^2}dx^2 + \frac{2d^2z}{dx\,dy}dx\,dy + \frac{d^2z}{dy^2}dy^2$, it will prove to be the same as that which would arise from differentiating $u = 0$ twice, on the supposition that z is a function of two independent quantities, x and y, which flow

* The number of dots, when there are more than one, are only intended to mark the number of *independent* principals.

uniformly. This ought to be the result, for $\ddot{d}u$ is a function of $x, y, z,$ and z is a function of $x, y,$ consequently $\ddot{d}u$ is a function of x, y, and therefore $\frac{d^2u}{dx^2}dx^2 + \frac{2d^2u}{dxdy}dxdy + \frac{d^2u}{dy^2}dy^2 = \ddot{d}u$.

35. *If there are two functions and three variables, the functions may be each developed in a series ascending by the powers of either of the variables, and equations may be derived expressing the relation between the fluxional coefficients of two of the variables considered as functions of the third.*

Let $u = 0$ and $v = 0$ be the functions; $x, y, t,$ the variables; and let it be required to develope u and v in terms of g, the increment of t.

(Art. 20.) $$\left.\begin{array}{l} \text{U} = u + \frac{du}{dx}\frac{h}{1} + \frac{du}{dy}\frac{k}{1} + \frac{du}{dt}\frac{g}{1} \\ \quad + \&\text{c.} \end{array}\right\} = 0$$

$$\left.\begin{array}{l} \text{V} = v + \frac{dv}{dx}\frac{h}{1} + \frac{dv}{dy}\frac{k}{1} + \frac{dv}{dt}\cdot\frac{g}{1} \\ \quad + \&\text{c.} \end{array}\right\} = 0$$

But, since there are three indeterminates and two equations, by elimination x and y may each be expressed in terms of t alone, or may be considered as functions of t; hence we have

$$\left.\begin{array}{l} h = \frac{dx}{dt}\frac{g}{1} + \frac{d^2x}{dt^2}\frac{g^2}{1.2} + \&\text{c.} \\ \text{and } k = \frac{dy}{dt}\frac{g}{1} + \frac{d^2y}{dt^2}\frac{g^2}{1.2} + \&\text{c.} \end{array}\right\} \text{which, substituted in the}$$

above equations, will give the developements of U and of V in series ascending by the powers of g.

The coefficient of g in the developement of U is $\frac{du}{dt} + \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt}$, which therefore $= \frac{\dot{d}u}{dt} = 0$. For the same reason $\frac{\dot{d}v}{dt} = \frac{dv}{dt} + \frac{dv}{dx}\frac{dx}{dt} + \frac{dv}{dy}\frac{dy}{dt} = 0$.

The succeeding fluxional coefficients may be found either by arranging the terms of the series according to the powers of g, or by successive differentiations.

It is obvious that the same method may be extended to any number of functions, provided that the number of variables is greater by unity than the number of equations.

36. *If the number of variables exceeds the number of equations* + 1, *there is more than one independent principal, and there may be derived from each of the equations more than one fluxional equation of the first order.*

For instance, let $u = 0$ and $v = 0$ contain five variables, z, y, x, t, s; then, since there are only two equations, three of these variables may be considered as arbitrary or independent quantities, and the remaining two as functions of them.

Let z and y be each considered as functions of s, t, x; then differentiating $u = 0$ partially with respect to s, t, and x, we have

$$\therefore \frac{du}{ds} = 0 = \frac{du}{ds} + \frac{du}{dy}\frac{dy}{ds} + \frac{du}{dz}\frac{dz}{ds}$$

$$\therefore \frac{du}{dt} = 0 = \frac{du}{dt} + \frac{du}{dy}\frac{dy}{dt} + \frac{du}{dz}\frac{dz}{dt}$$

$$\therefore \text{and } \frac{du}{dx} = 0 = \frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} + \frac{du}{dz}\frac{dz}{dx}.$$

Similar equations may be derived from $v = 0$.

Cor. 1. Multiply the first equation by ds, the second by dt, and the third by dx; add them together; and substitute in the result

$$\left.\begin{aligned} dy &= \frac{dy}{ds}ds + \frac{dy}{dt}dt + \frac{dy}{dx}dx \\ \text{and } dz &= \frac{dz}{ds}ds + \frac{dz}{dt}dt + \frac{dz}{dx}dx \end{aligned}\right\}\text{and we shall have}$$

$$\therefore du = 0 = \frac{du}{ds}ds + \frac{du}{dt}dt + \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz.$$

Cor. 2. Since the partial fluxional coefficients contained in these equations are all functions of z, y, x, t, s, they may be again differentiated partially with respect to either of the independent principals; and by repeating the operation on

either of the equations $u = 0$, $v = 0$, there will arise a succession of equations which show the relation between the fluxional coefficients of the dependent variables considered as functions of the independent.

This method may be readily extended to any number of functions containing any number of variables. The conditions of the question show the number of the variables which may be assumed to be independent, and we must differentiate successively on the hypothesis that the remaining variables are implicit functions of these.

Functions also of two or more variables may be differentiated on other hypotheses of the variables, which we shall not here consider; but shall proceed to show certain advantages that may be derived from differentiation combined with the process of elimination.

37. *Fluxional equations may be derived from their primitive by successive differentiations combined with eliminations.*

As this will form the subject of Vol. ii. Ch. 2. we shall give but one example.

Let $y + ax + b = 0$, then considering x as the independent variable, we have $\frac{dy}{dx} + a = 0$, and eliminating x,

$$y - \frac{dy}{dx}x + b = 0.$$

In this example we have two fluxional equations, both of which belong to the same primitive; and it is evident that, in general, by the process of elimination, the same primitive may belong to a great variety of fluxional equations of the same order, exclusive of those which may be derived from differentiating on different hypotheses of the variables.

38. *By successive differentiations, the irrational and transcendental functions of an equation may be eliminated.*

Ex. 1. $y = (a^2 + x^2)^{\frac{m}{n}}$.

$dy = \frac{m}{n}(a^2 + x^2)^{\frac{m}{n}-1} 2xdx = \frac{m(a^2+x^2)^{\frac{m}{n}} 2xdx}{n(a^2+x^2)}$ $\therefore$ eliminating $(a^2 + x^2)^{\frac{m}{n}}$, we have $dy = \frac{2myxdx}{n(a^2 + x^2)}$, in which $(a^2 + x^2)^{\frac{m}{n}}$ does not appear.

In the same manner, if the equation contain n irrational functions, by differentiating n times, and placing the results under a proper form, we shall have n equations from which the irrational functions may be eliminated.

Ex. 2. $lx + ly + z = 0$

$\therefore \ \dfrac{dx}{x} + \dfrac{dy}{y} + dz = 0$, an equation free from transcendentals.

Ex. 3. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$.

$y = \dfrac{e^{2x} + 1}{e^{2x} - 1} \ \therefore \ e^{2x} = \dfrac{y+1}{y-1}$ or $2x = l\dfrac{y+1}{y-1}$ & $dx = \dfrac{dy}{1-y^2}$.

Ex. 4. $y = l.\dfrac{e^x + e^{-x}}{2}$.

$\dfrac{dy}{dx}$ or $p = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x} - 1}{e^{2x} + 1} \ \therefore \ e^{2x} = -\dfrac{p+1}{p-1} \ \ldots\ldots$

$=$ (vid. 2. 16.) $\dfrac{1+p}{1-p} \ \therefore \ 2x = l(1+p) - l(1-p) \ \therefore$

$2 = \dfrac{q}{1+p} + \dfrac{q}{1-p} = \dfrac{2q}{1-p^2}$ or $q + p^2 - 1 = 0$ i. e. $\dfrac{d^2y}{dx^2}$ $+ \dfrac{dy^2}{dx^2} - 1 = 0$, an equation which is free from transcendentals.

Ex. 5. $y = \sin.x$.

$dx = \dfrac{dy}{\sqrt{1-y^2}}$ and $\dfrac{dy^2}{dx^2} = 1 - y^2$.

Ex. 6. $y = a \sin.^{-1}\dfrac{x}{a}$.

$dy = \dfrac{adx}{\sqrt{a^2 - x^2}}$ and $\dfrac{dy^2}{dx^2} = \dfrac{a^2}{a^2 - x^2}$.

Ex. 7. $y = \cot.^{-1} x^2 \ \therefore \ \dfrac{dy}{dx} = -\dfrac{2y^{\frac{1}{2}}}{1+x^2}$.

Ex. 8. $y = x.e^{\tan.^{-1}x} \ \therefore \ \dfrac{dy}{dx} = \dfrac{y}{x}.\dfrac{1+x+x^2}{1+x^2}$.

Ex. 9. $y = \tan.^{-1}\dfrac{x+y}{a} \ \therefore \ \dfrac{dy}{dx} = \dfrac{a^2}{(x+y)^2}$.

Ex. 10. $y = a \sin. x + b \cos. x$.

Differentiating twice, and eliminating, there results $\frac{d^2y}{dx^2} + y = 0$.

Ex. 11. $u = x^{yz} \therefore \frac{du}{dx} x - uyz = 0$.

The application of this process to the developement of functions has been already shown Art. 11.

39. *By differentiation and elimination there may be made to disappear from an equation of three variables functions whose form is unknown and indeterminate.*

Let the equation be $z = f(x, y)$, where x and y are independent quantities.

By substitution, let $z = ft$, then t is a function of (x, y); and consequently, differentiating partially, we have (3.8)

$$\left.\begin{array}{l} \dfrac{dz}{dx} = \dfrac{d.ft}{dt}\dfrac{dt}{dx} \\ \dfrac{dz}{dy} = \dfrac{d.ft}{dt}\dfrac{dt}{dy} \end{array}\right\} \text{therefore, eliminating } \frac{dft}{dt}, \text{ we have } \frac{dt}{dy} \times \frac{dz}{dx}$$

$= \frac{dt}{dx} \times \frac{dz}{dy}$, a fluxional equation in which the coefficients $\frac{dt}{dy}$ and $\frac{dt}{dx}$ are independent of f.

We have supposed the equation to be of the form $z = f(x, y)$; but the same result will be obtained, if we suppose that it is of the form $z = \text{F}(x, y, f(x, y))$ where $f(x, y)$ is combined in any manner with the variables.

For, let $u = 0$ represent the equation $z = \text{F}(x, y, ft) =$, by substitution, $\text{F}(x, y, s)$, then differentiating u partially as a function containing two implicit functions z and s, whose principal variables are x and y, we have

$$\left.\begin{array}{l} \dfrac{\ddot{d}u}{dx} = 0 = \dfrac{du}{dx} + \dfrac{du}{dz}\dfrac{dz}{dx} + \dfrac{du}{ds}\dfrac{ds}{dx} \\ \dfrac{\ddot{d}u}{dy} = 0 = \dfrac{du}{dy} + \dfrac{du}{dz}\dfrac{dz}{dy} + \dfrac{du}{ds}\dfrac{ds}{dy} \end{array}\right\}$$

But since $s = ft$ and t is a function of (x, y), we have

$$\left.\begin{aligned}\frac{ds}{dx} &= \frac{ds}{dt}\frac{dt}{dx}\\ \frac{ds}{dy} &= \frac{ds}{dt}\frac{dt}{dy}\end{aligned}\right\} \text{therefore, by substitution,}$$

$$\left.\begin{aligned}\frac{\dot{d}u}{dx} = 0 &= \frac{du}{dx} + \frac{du}{dz}\frac{dz}{dx} + \frac{du}{ds}\frac{ds}{dt}\frac{dt}{dx}\\ \frac{\dot{d}u}{dy} = 0 &= \frac{du}{dy} + \frac{du}{dz}\frac{dz}{dy} + \frac{du}{ds}\frac{ds}{dt}\frac{dt}{dy}\end{aligned}\right\} \text{from which}$$

$\frac{ds}{dt}$ may be eliminated; also the function s in $\frac{du}{ds}$ may be eliminated by means of the equation $z = \text{F}(x, y, s)$, and there will result an equation between $\frac{dz}{dx}$ and $\frac{dz}{dy}$, which is independent of f.

Ex. 1. $z = f(x^2+y^2)$.

Assume $t = x^2 + y^2$, then the fluxional equation derived by the process pointed out in the former part of the article is $y\frac{dz}{dx} - x\frac{dz}{dy} = 0$, which is independent of f; or the equation obtains whether we suppose $z = (x^2 + y^2)^n$ or $= a^{x^2+y^2}$ or $= l(x^2 + y^2)$ or any function whatever of x^2+y^2.

Ex. 2. $z = f(ax + by)$.

Let $t = ax + by$, then the resulting equation is $b\frac{dz}{dx} - a\frac{dz}{dy} = 0$, which expresses the relation between $\frac{dz}{dx}$ and $\frac{dz}{dy}$, whether $z = \sqrt{ax+by}$ or $= \sin.(ax+by)$, or any function of $ax+by$.

PRAXIS.

1. $z = f\frac{y^2-x^2}{x} \quad \therefore \frac{dz}{dx} + \frac{x^2+y^2}{2xy}\frac{dz}{dy} = 0.$

2. $z = \frac{cy}{a} + f(ax - by) \therefore b\frac{dz}{dx} + a\frac{dz}{dy} - c = 0.$

3. $z = \frac{x^2}{2} + f(y + lx) \therefore x\frac{dz}{dx} - \frac{dz}{dy} - x^2 = 0.$

It was shown, Articles 27 and 37, that from the same primitive there may in some cases be derived a great variety of fluxional equations of the same order; and it appears from this article that to the same fluxional equation between three variables, one of which is considered as an implicit function of the other two, there may belong an infinite number of primitives.

We shall not for the present extend these considerations to the higher orders of fluxional equations.

40. *The preceding article furnishes a criterion by which we can ascertain whether any proposed quantity consisting of two variables is a function of any other proposed quantity.*

For if one quantity is a function of another, they will both satisfy the same fluxional equation.

Thus, to determine whether $x^4 + 2x^2y^2 + y^4$ is a function of $x^2 + y^2$, we find, substituting $z = x^4 + 2x^2y^2 + y^4$, that it satisfies the equation $y\frac{dz}{dx} - x\frac{dz}{dy} = 0$; so also $z = a^2x^2 + 2abxy + b^2y^2$ satisfies the equation $b\frac{dz}{dx} - a\frac{dz}{dy}$; hence we conclude that $x^4 + 2x^2y^2 + y^4$ and $a^2x^2 + 2abxy + b^2y^2$ are functions of $x^2 + y^2$ and of $ax + by$ respectively.

41. *Lagrange's Theorem.*

Let $z = x + yfz$, where x and y are independent quantities; also, suppose $u = \phi z$, then shall

$$u = \phi z = \phi x + \frac{y}{1}.\ \phi' x f x + \frac{y^2}{1.2}(\phi' x f x^2)' + \frac{y^3}{1.2.3}(\phi' x f x^3)'' + \&c.$$

First, suppose $\phi z = z$, and let it be required to prove that

$$z = x + \frac{y}{1}.fx + \frac{y^2}{1.2}(fx^2)' + \frac{y^3}{1.2.3}(fx^3)'' + \&c.:$$

assume $z = x + \text{A}y + \text{B}y^2 + \text{C}y^3 + \text{D}y^4 + \&c.$; where the first term is x, because when $y = 0$, $z = x$.

Also, $\text{A} = fx$; for diminishing y without limit, $z = x$, and therefore $fz = fx$, or the series must be such as to answer

the condition $z = x + yfz$ when y is diminished without limit (1. 44 Cor. 2 or 3. 2).

Now differentiate the original equation partially, and first with respect to x; then, since z is a function of x and y,
$\frac{dfz}{dx}dx = \frac{dfz}{dz}\frac{dz}{dx}dx = f'z\, z'$; similarly $\frac{dfz}{dy}dy = f'z\, z_{,}$;
hence we have

$$\left.\begin{aligned} z' &= 1 + yf'z\, z' \\ z_{,} &= fz + yf'z\, z_{,} \end{aligned}\right\}$$ therefore, eliminating $yf'z$, $\frac{z'-1}{z'} = \frac{z_{,}-fz}{z_{,}}$

or $1 - \frac{1}{z'} = 1 - \frac{fz}{z_{,}}$, therefore $z_{,} = z'fz = z'.\frac{z-x}{y}$ or

$z_{,}\,y = z'(z-x)$. (α)

If then we find the values of $z_{,}$ and of z' from the series assumed for z and divide by y, we shall have from (α)

$$\text{A} + 2\text{B}y + 3\text{C}y^2 + 4\text{D}y^3 + \&\text{c.} \ldots\ldots$$
$$= \{1 + \text{A}'y + \text{B}'y^2 + \text{C}'y^3 + \text{D}'y^4 + \&\text{c.}\}\{\text{A} + \text{B}y + \text{C}y^2 + \text{D}y^3 + \&\text{c.}\}$$

or

$$\left.\begin{aligned} &\text{A} \\ &+(\text{B}+\text{AA}')y \\ &+(\text{C}+\text{A}'\text{B}+\text{AB}')y^2 \\ &+(\text{D}+\text{A}'\text{C}+\text{BB}'+\text{AC}')y^3 \\ &+\&\text{c.} \end{aligned}\right\} = \text{A} + 2\text{B}y + 3\text{C}y^2 + 4\text{D}y^3 + \&\text{c.}$$

and equating coefficients (Alg. 346),

$$\begin{array}{r|l} & \text{or} \\ & \text{A} = fx \\ \text{B} + \text{AA}' = 2\text{B} & \text{B} = \text{AA}' \\ \text{C} + \text{A}'\text{B} + \text{AB}' = 3\text{C} & \text{C} = \tfrac{1}{2}\{\text{AB}' + \text{A}'\text{B}\} \\ \text{D} + \text{A}'\text{C} + \text{BB}' + \text{AC}' = 4\text{D} & \text{D} = \tfrac{1}{3}\{\text{AC}' + \text{BB}' + \text{A}'\text{C}\} \\ \&\text{c.} = \&\text{c.} & \&\text{c.} = \&\text{c.} \end{array}$$

Hence $\text{A} = fx$

$$\text{B} = \text{AA}' = fx\, f'x = \tfrac{1}{2}(fx^2)'$$

$$\text{C} = \tfrac{1}{2}(\text{AB})' = \tfrac{1}{2}(fx^2 f'x)' = \frac{1}{2.3}(fx^3)''$$

$$\text{D} = \tfrac{1}{3}\left(\text{AC} + \frac{\text{B}^2}{2}\right)' = \tfrac{1}{3}\left(fx.\frac{(fx^3)''}{2.3} + \frac{fx^2 f'x^2}{2}\right)'$$

$$= \tfrac{1}{3}\left(fx.\frac{(fx^2 f'x)'}{2} + f'x.\frac{fx^2 f'x}{2}\right)'$$

$$= \tfrac{1}{3}\,\frac{(fx^3 f'x)''}{2} = \frac{1}{2.3.4}(fx^4)'''$$

&c. = &c.

Next, let $u=\phi z$, then differentiating partially as before

$$\left.\begin{array}{l} u'=\phi' z\, z' \\ u_{\prime}=\phi' z\, z_{\prime} \end{array}\right\} \text{where } \phi' z=\frac{d\phi z}{dz}.$$

Hence, eliminating $\phi' z$, $\dfrac{u'}{u_{\prime}}=\dfrac{z'}{z_{\prime}}$; but from equation ($\alpha$) $\dfrac{z'}{z_{\prime}}=\dfrac{y}{z-x}$, therefore, by substitution, $\dfrac{u'}{u_{\prime}}=\dfrac{y}{z-x}$ and $u'(z-x)=yu_{\prime}$. (β)

Now assume $u=\text{P}+\text{Q}y+\text{R}y^2+\text{S}y^3+\&c.$, P, Q, R, &c. being sole functions of x; therefore

$$\left.\begin{array}{l} u'=\text{P}'+\text{Q}'y+\text{R}'y^2+\&c. \\ u_{\prime}=\text{Q}+2\text{R}y+3\text{S}y^2+\&c. \end{array}\right\} \begin{array}{l}\text{Substitute these in equation } (\beta), \\ \text{and dividing by } y, \text{ we have}\end{array}$$

$$\left\{\text{P}'+\text{Q}'y+\text{R}'y^2+\&c.\right\}\left\{fx+\frac{y}{1.2}(fx^2)'+\frac{y^2}{1.2.3}(fx^3)''+\&c.\right\}$$

$=u_{\prime}=\text{Q}+2\text{R}y+3\text{S}y^2+\&c.$; & equating coefficients, we have

$$\text{Q}=\text{P}'fx$$

$$2\text{R}=\text{Q}'fx+\frac{\text{P}'}{2}(fx^2)'$$

$$3\text{S}=\text{R}'fx+\frac{\text{Q}'}{2}(fx^2)'+\frac{\text{P}'}{2.3}(fx^3)''$$

$$\&c.=\&c.$$

But $\text{P}=\phi x$; for $\text{P}=(u)=(\phi z)=\phi x$; therefore

$$\text{Q}=\text{P}'fx=\phi' x\, fx$$

$$2\text{R}=\text{Q}'fx+\frac{\text{Q}}{2fx}\times 2fx\,f'x$$

$$=\text{Q}'fx+\text{Q}f'x$$

$$=(\text{Q}fx)'=(\phi' x\, fx^2)', \text{ therefore } \text{R}=\tfrac{1}{2}(\phi' x\, fx^2)'.$$

Similarly it may be shown that $\text{S}=\dfrac{1}{2.3}(\phi' x\, fx^3)''$ and so on.

Hence ϕz, or $u=\phi x+\dfrac{y}{1}\phi' x\, fx+\dfrac{y^2}{1.2}(\phi' x\, fx^2)'$ .....

$+\dfrac{y^3}{1.2.3}(\phi' x\, fx^3)''+\&c.$

Cor. 1. If $y=1$, $u=\phi x+\phi' x\, fx+\dfrac{1}{1.2}(\phi' x\, fx^2)'$ .....

$+\dfrac{1}{1.2.3}(\phi' x\, fx^3)''+\&c.$

Cor. 2. If $y=h$ and $fz=1$, the theorem becomes
$u=\varphi(x+h)=\varphi x+\varphi' x\frac{h}{1}+\varphi'' x\frac{h^2}{1.2}+$ &c., which is Taylor's theorem.

Lagrange has also deduced this very useful theorem from Maclaurin's. Fonct. Anal. p. 149.

42. We offer the following as a more direct investigation of the above developement, in which it is not necessary to assign a particular value to the variable in order to give to the coefficients the required form.

$u=\varphi z=\varphi(x+yfz)$, by Taylor's Theorem . . . .

$$=\varphi x+\varphi' x.\frac{yfz}{1}+\varphi'' x.\frac{y^2fz^2}{1.2}+\varphi''' x.\frac{y^3fz^3}{1.2.3}+\text{\&c.}$$

$$=\varphi x+\frac{y.\varphi' x}{1}\left\{fx+f'x.\frac{yfz}{1}+f''x.\frac{y^2fz^2}{1.2}+\text{\&c.}\right\}$$
$$+\frac{y^2}{1.2}.\varphi'' x\left\{fx^2+2fxf'x.\frac{yfz}{1}+\text{\&c.}\right\}$$
$$+\frac{y^3}{1.2.3}.\varphi''' x\left\{fx^3+\text{\&c.}\right\}$$

$$=\varphi x+y.\varphi' xfx+y^2\left\{\begin{array}{l}\varphi' xf'x\,(fx+f'x.\frac{yfz}{1}+\text{\&c.})\\+\frac{\varphi'' xfx^2}{1.2}\end{array}\right.$$
$$+y^3\left\{\begin{array}{l}\frac{\varphi' xf''x+\text{\&c.}}{1.2}(fx^2+\text{\&c.})\\+\varphi'' xfxf'x\,(fx+\text{\&c.})\\+\frac{\varphi''' x}{1.2.3}fx^3\end{array}\right.$$

$$=\varphi x+y.\varphi' xfx+\frac{y^2}{1.2}(2\varphi' xfxf'x+\varphi'' xfx^2)$$
$$+\frac{y^3}{1.2.3}\left(\begin{array}{l}6\varphi' xf'x^2fx+3\varphi' xf''xfx^2\\+6\varphi'' xfx^2f'x+\varphi''' xfx^3\end{array}\right)+\text{\&c.}$$

$$=\phi x+y.\phi' xfx+\frac{y^2}{1.2}(\phi' xfx^2)'+\frac{y^3}{1.2.3}(\phi' xfx^3)''+\text{\&c.}$$

Cor. 1. When $y=0$, $z=x$; hence the theorem may be put under the form

$$u = \phi x + \frac{y}{1}\phi' x\,(fz) + \frac{y^2}{1.2}(\phi' x(fz^2))' + \frac{y^3}{1.2.3}(\phi' x\,(fz^3))''$$

+&c. where (u), (fz).... are the values of u, fz, ... when $y = 0$.

Cor. 2. Let $z = F(x + yfz)$; and suppose that $\phi(F) = \psi$, $f(F) = \chi$; then expanding $u = \psi(x + yfz)$ as in the article, there will result

$$u = \psi x + \frac{y}{1}\psi' x\,\chi x + \frac{y^2}{1.2}(\psi' x\,\chi x^2)' + \frac{y^3}{1.2.3}(\psi' x \chi x^3)'' + \&c.$$

Cor. 3. The preceding theorem may be put under the form

$$u = \psi x + \frac{y}{1}\psi' x(\chi z) + \frac{y^2}{1.2}(\psi' x(\chi z^2))' + \frac{y^3}{1.2.3}(\psi' x(\chi z^3))'' + \&c.$$

Of this theorem Laplace has given a demonstration remarkable for its elegance, in the Mecanique Cèleste, Tom. 1. L. 2. Num. 21.

Examples.

Ex. 1. Given the quadratick $a - bz + cz^2 = 0$; required to develope z.

$$z = \frac{a}{b} + \frac{cz^2}{b} \therefore \text{ here}$$

$$\left.\begin{array}{l} x = \dfrac{a}{b} \\ fz = \dfrac{cz^2}{b} \\ \therefore fx = \dfrac{cx^2}{b} \end{array}\right| \begin{array}{l} \therefore fx^2 = \dfrac{c^2}{b^2}x^4 \therefore (fx^2)' = \dfrac{4c^2}{b^2}x^3; \ \& fx^3 = \dfrac{c^3x^6}{b^3} \\ \therefore (fx^3)' = \dfrac{6c^3x^5}{b^3} \therefore (fx^3)'' = \dfrac{5.6\,c^3x^4}{b^3}. \\ \text{Similarly } (fx^4)''' = 6.7.8\,\dfrac{c^4x^5}{b^4}\ ; \ \&c. = \&c. \end{array}$$

Hence (40. Cor. 1.)

$$z = \frac{a}{b} + \frac{cx^2}{b} + \frac{4c^2x^3}{1.2\,b^2} + \frac{5.6c^3x^4}{1.2.3\,b^3} + \&c.$$

$$= \frac{a}{b}\left\{1 + \frac{ac}{b^2} + \frac{4a^2c^2}{1.2\,b^4} + \frac{5.6\,a^3c^3}{1.2.3\,b^6} + \frac{6.7.8\,a^4c^4}{1.2.3.4\,b^8} + \&c.\right\},$$

which is the same series as that which would arise from developing by the binomial theorem $\dfrac{b}{2c} - \dfrac{\sqrt{b^2 - 4ac}}{2c}$ the *least* root of the proposed equation.

The other root may be deduced by changing the form of the equation: thus, since $z=\frac{b}{c}-\frac{a}{cz}$, we have

$$\begin{array}{l|l} x=\frac{b}{c} & \therefore fx^2=\frac{a^2}{c^2x^2}\ \therefore (fx^2)'=-\frac{2a^2}{c^2x^3}\text{; and } fx^3= \\ fz=-\frac{a}{cz} & -\frac{a^3}{c^3x^3}\ \therefore (fx^3)''=-\frac{3.4\,a^3}{c^3x^5}\text{; \&c.} = \text{\&c.} \\ \therefore fx=-\frac{a}{cx} & \end{array}$$

$$\text{Hence } z=\frac{b}{c}-\frac{a}{cx}-\frac{2a^2}{1.2\,c^2x^3}-\frac{3.4\,a^3}{1.2.3c^3x^5}-\text{\&c.}$$

$$=\frac{b}{c}-\frac{a}{b}\left\{1+\frac{ac}{b^2}+\frac{2a^2c^2}{b^4}+\text{\&c.}\right\}\text{ which}$$

is the same as $\frac{b}{2c}+\frac{\sqrt{b^2-4ac}}{2c}$.

Ex. 2. Given the cubick $z^3-qz+r=0$; required to develope z.

$$z=\frac{r}{q}+\frac{z^3}{q}\ \therefore \text{ here}$$

$$\begin{array}{l|l} x=\frac{r}{q} & \\ fz=\frac{1}{q}.z^3 & \therefore (fx^2)'=\frac{6}{q^2}x^5\text{; \& } (fx^3)''=\frac{8.9}{q^3}x^7\text{; \&c.}=\text{\&c.;} \\ \therefore fx=\frac{1}{q}.x^3 & \end{array}$$

$$\text{hence } z=\frac{r}{q}+\frac{1}{q}\,\frac{r^3}{q^3}+\frac{6}{1.2q^2}\,\frac{r^5}{q^5}+\frac{8.9}{1.2.3q^3}\,\frac{r^7}{q^7}+\text{\&c.}$$

$$=\frac{r}{q}\left\{1+\frac{r^2}{q^3}+\frac{6r^4}{1.2q^6}+\frac{8.9r^6}{1.2.3\,q^9}+\text{\&c.}\right\}\text{ which}$$

is the *least* root of the proposed equation.

Ex. 3. Given $ay^3+y^3z-az^3=0$; required z in a series in terms of y.

$$z=-a+\frac{a}{y^3}z^3.$$

$$\begin{array}{r|l} \text{Here } x = -a & fx = x^3 = (-a)^3;\ (fx^2)' = 6.(-a)^5; \\ y = \dfrac{a}{y^3} & (fx^3)'' = 8.9\,(-a)^7;\ \&c. = \&c. \\ fz = z^3 & \end{array}$$

$$\therefore z = -a + \frac{a}{y^3}(-a)^3 + \frac{a^2}{1.2\,y^6} \times 6(-a)^5 + \frac{a^3}{1.2.3\,y^9} \ldots \times 8.9(-a)^7 + \&c.$$

$$= -a - \frac{a^4}{y^3} - \frac{6a^7}{1.2y^6} - \frac{8.9a^{10}}{1.2.3\,y^9} - \&c.$$

Ex. 4. Given $y^3 - 2xy^2 + x^2y - a^3 = 0$; required y in terms of x.

By the solution of a quadratick, $x = y \pm \dfrac{a^{\frac{3}{2}}}{y^{\frac{1}{2}}}$, therefore we have $y = x \mp \dfrac{a^{\frac{3}{2}}}{y^{\frac{1}{2}}} =$ (40. Cor. 1.) $x \mp \dfrac{a^{\frac{3}{2}}}{x^{\frac{1}{2}}} - \dfrac{1}{1.2}\,\dfrac{a^3}{x^2}$ $\mp \dfrac{1}{1.2.3}\cdot\dfrac{3.5}{2.2}\,\dfrac{a^{\frac{9}{2}}}{x^{\frac{7}{2}}} - \&c.$

Also, the equation may be put under the form

$y = \dfrac{a^3}{x^2} + \left(\dfrac{2y^2}{x} - \dfrac{y^3}{x^2}\right) \therefore$ by the theorem

$$\left.\begin{aligned} y &= \frac{a^3}{x^2} + \left(\frac{2}{x}\,\frac{a^6}{x^4} - \frac{1}{x^2}\,\frac{a^9}{x^6}\right) + \&c. \\ &= \frac{a^3}{x^2} + \frac{2a^6}{x^5} - \frac{7a^9}{x^8} + \&c. \end{aligned}\right\} \text{(Vid. 3. 10. Ex. 4.)}$$

Ex. 5. Given $1 - z + az = 0$; required to develope lz.

$z = 1 + az \therefore$ here

$$\begin{array}{l|l|l} x = 1 & & \\ fz = z & fx = x & \phi' x = \dfrac{1}{x} \therefore \phi' x f x^2 = x \therefore (\phi' x f x^2)' = 1; \\ \phi z = lz & \phi x = lx & \end{array}$$

also $(\phi' x f x^3) = x^2 \therefore (\phi' x f x^3)'' = 1.2.$

Similarly $(\varphi' x f x^4)''' = 1.2.3$; &c. = &c.; hence
$lz = 0 + \frac{a}{1} + \frac{a^2}{2} + \frac{a^3}{3} + \frac{a^4}{4} + \frac{a^5}{5} +$ &c., which is therefore the *h. l.* of $\frac{1}{1-a}$.

From the same equation, by means of the theorem, there may be developed z^n, sin.z, $l(2z-1)$ or any function of z.

Ex. 6. Given $a - bz + cz^m = 0$; required to develope z^n.

$z = \frac{a}{b} + \frac{cz^m}{b} =$, by substitution, $e + yz^m$ $\therefore$ here

$$\therefore \quad \begin{array}{l|l|l} fz = z^m & fx = x^m = e^m & \therefore \varphi' x = ne^{n-1}; \ \& \ \varphi' x f x^2 = ne^{2m+n-1} \\ \varphi z = z^n & \varphi x = x^n = e^n & \therefore (\varphi' x f x^2)' = n(2m+n-1)\ e^{2m+n-2}; \end{array}$$

and $\varphi' x f x^3 = ne^{3m+n-1}$ $\therefore$ $(\varphi' x f x^3)' = n(3m+n-1)\ e^{3m+n-2}$ $\therefore$ $(\varphi' x f x^3)'' = n(3m+n-1)\ (3m+n-2)\ e^{3m+n-3}$; &c. = &c.; hence, we have

$$z^n = e^n + \frac{y}{1}.\ ne^{m+n-1} + \frac{y^2}{1.2}\ n(2m+n-1)\ e^{2m+n-2} \ \ldots\ldots$$
$$+ \frac{y^3}{1.2.3}\ n(3m+n-1)\ (3m+n-2)\ e^{3m+n-3} + \text{\&c.}$$
$$= \frac{a^n}{b^n}\left\{1 + \frac{nca^{m-1}}{b^m} + \frac{n(2m+n-1)c^2a^{2m-2}}{1.2b^{2m}} \ \ldots\ldots\ldots \right.$$
$$\left. + \frac{n(3m+n-1)\ (3m+n-2)c^3a^{3m-3}}{1.2.3b^{3m}} + \text{\&c.}\right\}.$$

The reader may consult "Mémoires de l'Acad. de Berlin for the year 1768, page 275," and the "Traité de la Resolution des Equations numeriques, note 11," in which Lagrange has shown that the developement given by this theorem corresponds to the *least* of the roots of the proposed equation.

Ex. 7. Given $z^2 - \frac{2z}{e} + 1 = 0$; required z^n.

$z = \frac{e}{2} + \frac{ez^2}{2}$.

$$\begin{array}{l|l|l} \therefore\ x = \frac{e}{2} & & \therefore\ z^n = x^n + \frac{ne}{2} x^{n+1} + \frac{ne^2}{2^3}(x^{n+3})' \ \ldots \\ fz = \frac{ez^2}{2} & fx = \frac{ex^2}{2} & \qquad + \frac{ne^3}{3.2^4}(x^{n+5})'' + \text{\&c.} \\ \varphi z = z^n & \varphi x = x^n & \end{array}$$

$$= \frac{c^n}{2^n}\left\{1 + \frac{ne^2}{2^2} + \frac{n.(n+3)}{1.2}\frac{e^4}{2^4} + \frac{n.(n+4)\ (n+5)}{1.2.3}\frac{e^6}{2^6} + \&c.\right\}$$

If $n = -1, \frac{1}{z} = \frac{2}{e}\left\{1 - \frac{e^2}{2^2} + \frac{e^4}{2^4} - \frac{3.4}{2.3}\frac{e^6}{2^6} + \&c.\right\}$ which, since $z = \frac{1 \pm \sqrt{1-e^2}}{e}$, is the developement of $\frac{e}{1-\sqrt{1-e^2}}$.

Ex. 8. Required to revert the series $x = az + bz^2 + cz^3 + ez^4 + \&c.$

$z = \frac{x}{a} - \frac{1}{a}\left\{bz^2 + cz^3 + ez^4 + \&c.\right\}$ $\therefore$ here

$x = \frac{x}{a}$	$\therefore fx = bx^2 + cx^3 + \&c.$ where $x = \frac{x}{a}$,
$y = -\frac{1}{a}$	$\therefore fx^2 = (bx^2 + cx^3 + \&c.)^2$ $\therefore$ $(fx^2)' =$
$fz = bz^2 + cz^3 + \&c.$	$2(bx^2 + cx^3 + \&c.)\ \{2bx + 3cx^2 + \&c.\}$;

$\therefore (fx^3)' = 3(bx^2 + cx^3 + \&c.)^2\ \{2bx + 3cx^2 + \&c.\}$;
$\therefore (fx^3)'' = 2.3(bx^2 + cx^3)\ \{2bx + 3cx^2\}^2 + 3(bx^2 + cx^3)^2 \ldots$
$\{2b + 2.3cx\}$; &c. = &c.; hence

$$z = \frac{x}{a} - \frac{1}{a}\left\{\frac{bx^2}{a^2} + \frac{cx^3}{a^3} + \frac{ex^4}{a^4}\right\}$$
$$+ \frac{1}{a^2}\left\{\frac{bx^2}{a^2} + \frac{cx^3}{a^3}\right\}\left\{\frac{2bx}{a} + \frac{3cx^2}{a^2}\right\}$$
$$- \frac{1}{a^3}\left\{\frac{bx^2}{a^2} + \frac{cx^3}{a^3}\right\}\left\{\frac{2bx}{a} + \frac{3cx^2}{a^2}\right\}^2$$
$$+ \frac{1}{1.2.3^2}\left\{\frac{bx^2}{a^2} + \frac{cx^3}{a^3}\right\}^2\left\{2b + \frac{2.3cx}{a}\right\}$$
$$+ \&c.$$
$$= \frac{x}{a} - \frac{b^2}{a^3}x^2 + \frac{2b^2 - ac}{a^5}x^3 - \&c. \text{ (Alg. 351, Ex. 1.)}$$

Ex. 9. Given $az + bz^2 + cz^3 + \&c. = ax + \beta x^2 + \gamma x^3 + \&c.$; required z in terms of x.

$$z = \frac{1}{a}(ax + \beta x^2 + \gamma x^3 + \&c.) - \frac{1}{a}(bz^2 + cz^3 + \&c.)$$

$$\text{here } y = -\frac{1}{a} \qquad \therefore fx = \frac{b}{a^2}(\alpha x + \beta x^2 + \&c.)^2 \; . \; . \; . \; . \; .$$

$$x = \frac{1}{a}(\alpha x + \beta x^2 + \&c. \qquad + \frac{c}{a^3}(\alpha x + \beta x^2 + \&c.)^3; \text{ hence}$$

$$z = \frac{1}{a}(\alpha x + \beta x^2 + \gamma x^3 + \&c.$$

$$-\frac{1}{a}\left\{\frac{b}{a^2}(\alpha x + \beta x^2 + \&c.)^2 + \frac{c}{a^3}(\alpha x + \beta x^2 + \&c.)^3\right\}$$

$$+\frac{2}{a^2}\left\{\frac{b^2}{a^4}(\alpha x + \beta x^2 + \&c.)^3(\alpha + 2\beta x + \&c.) + \&c.\right\}$$

$$- \&c.$$

$$= \frac{\alpha}{a}x + \left(\frac{\beta}{a} - \frac{b\alpha^2}{a^3}\right)x^2 + \&c.$$

Ex. 10. Given $u + ph + \frac{qh^2}{1.2} + \frac{rh^3}{1.2.3} + \frac{sh^4}{1.2.3.4} + \&c. = 0$; required h in terms of u.

Here $h = -\frac{u}{p} - \frac{1}{p}\left\{\frac{qh^2}{1.2} + \frac{rh^3}{1.2.3} + \&c.\right\} \therefore$

$$x = -\frac{u}{p} \qquad \therefore (fx^2)' = 2\left(\frac{qx^2}{1.2} + \frac{rx^3}{1.2.3} + \&c.\right)$$

$$y = -\frac{1}{p} \qquad \left\{qx + \frac{rx^2}{1.2} + \&c.\right\};$$

$$fx = \frac{qx^2}{1.2} + \frac{rx^3}{1.2.3} + \&c. \qquad (fx^3)' = 3\left(\frac{qx^2}{1.2} + \frac{rx^3}{1.2.3} + \&c.\right)^2$$

$$\left\{qx + \frac{rx^2}{1.2} + \&c.\right\} \therefore (fx^3)'' = 2.3\left(\frac{qx^2}{1.2} + \frac{rx^3}{1.2.3}\right)\left(qx + \frac{rx^2}{1.2}\right)^2$$

$$+ 3\left(\frac{qx^2}{1.2} + \frac{rx^3}{1.2.3}\right)^2\left\{q + rx + \&c.\right\}; \; \&c. = \&c.; \text{ hence}$$

$$h = -\frac{u}{p} - \frac{1}{p}\left\{\frac{qx^2}{1.2} + \frac{rx^3}{1.2.3} + \&c.\right\}$$

$$+ \frac{1}{p^2}\left\{\frac{qx^2}{1.2} + \frac{rx^3}{1.2.3}\right\}\left\{qx + \frac{rx^3}{1.2}\right\}$$

$$- \frac{1}{p^2}\left\{\frac{qx^2}{1.2} \cdot q^2x^2 + \frac{q}{2} \cdot \frac{q^2x^4}{4}\right\}$$

$$= -\frac{u}{p} - \left\{\frac{q}{p^3}\frac{u^2}{1.2} + \frac{3q^2-pr}{p^5}\frac{u^3}{1.2.3} \cdots\cdots\cdots\right.$$
$$\left. + \frac{3.5q^3-2.5pqr+p^2s}{p^7}\frac{u^4}{1.2.3.4} + \text{\&c.}\right\}.$$

Cor. Since $q=\frac{dp}{dx}$, $r=\frac{dy}{dx}$, ... the series may be put under the form $h = -p^{-1}u + \frac{p^{-1}d.p^{-1}}{dx}\frac{u^2}{1.2} - \frac{p^{-1}d.p^{-1}d.p^{-1}}{dx^2}\frac{u^3}{1.2.3}$
$+ \frac{p^{-1}d.p^{-1}d.p^{-1}d.p^{-1}}{dx^3}\frac{u^4}{1.2.3.4}$ − &c.; a series which may be employed to great advantage in approximating to the roots of equations. Vid. Cambridge edition of Lacroix's Fluxional Calculus, vol. ii. p. 102.

Ex. 11. Given $r = \frac{a(1-e^2)}{1+e\cos.v}$; required r^n.

$r = a(1-e^2) - e\cos.v.\ r =$, by substitution, $x + y.r$;
$\phi x = x^n$ and $fx = x$; hence

$$r^n = x^n - \frac{y}{1}nx^n + \frac{y^2}{1.2}n(n+1)x^n - \frac{y^3}{1.2.3}n(n+1)(n+2)x^n + \text{\&c.}$$
$$= a^n.(1-e^2)^n\left\{1 - \frac{ne\cos.v}{1} + \frac{n.(n+1)}{1.2}e^2\cos.^2v - \text{\&c.}\right\}.$$

Required lr.

$$lr = lx - \frac{y}{1} + \frac{y^2}{2} - \frac{y^3}{3} + \frac{y^4}{4} - \text{\&c.}$$
$$= l.a(1-e^2) - \frac{e\cos.v}{1} + \frac{e^2\cos.^2v}{2} - \frac{e^3\cos.^3v}{3} + \text{\&c.}$$

Required to develope e.

$$e = -\frac{1}{\cos.v} + \frac{a}{r\cos.v}(1-e^2) =, \text{ by substitution,}$$
$$-x + y(1-e^2)\ \therefore$$
$$e = -x + \frac{y}{1}(1-x^2) + \frac{y^2}{1.2}.\,2^2(1-x^2)x \cdots\cdots\cdots$$
$$\cdots\cdots + \frac{y^3}{1.2.3}\left\{3.2^3(1-x^2)^2 - 2.3(1-x^2)^2\right\} + \text{\&c.};$$

but $1-x^2 = 1 - \frac{1}{\cos.^2v} = -\frac{\sin.^2v}{\cos.^2v}$; hence we have

$$e = -\frac{1}{\cos.v} - \frac{a\sin.^2 v}{r\cos.^3 v} - \frac{a^2}{1.2r^2}\,\frac{2^2\sin.^2 v}{\cos.^5 v} - \&c.$$

Ex. 12. Let $p = \cos.2x$, then shall

$$2\cos.nx = p^n - \frac{n}{1}p^{n-2} + \frac{n(n-3)}{1.2}p^{n-4} - \frac{n(n-4)(n-5)}{1.2.3}p^{n-6}$$
$$+ \frac{n(n-5)(n-6)(n-7)}{1.2.3.4}p^{n-8} - \&c.$$

It has been shown (2. 37) that if $2\cos.x = \alpha + \frac{1}{\alpha}$,

$$2\cos.nx = \alpha^n + \frac{1}{\alpha^n}.$$

First, to find α^n, we have $\alpha = 2\cos.x - \frac{1}{\alpha} = p - \frac{1}{\alpha}$ ∴

$$\alpha^n = p^n - np^{n-2} + \frac{n.(n-3)}{1.2}p^{n-4} - \frac{n.(n-4)(n-5)p^{n-6}}{1.2.3} + \&c.$$

Similarly $\frac{1}{\alpha^n} = p^{-n} + np^{-n-2} + \frac{n(n+3)}{1.2}p^{-n-4}$ . . . .

$$+ \frac{n(n+4)(n+5)}{1.2.3}p^{n-6} + \&c.$$

Hence $2\cos.nx$ = the sum of these series; but when n is an integer, if the first series be continued till the exponents of p become negative, each of the succeeding terms will be destroyed by a corresponding term in $\frac{1}{\alpha^n}$; thus,

$$\text{suppose } n = 1,\ \left.\begin{array}{l} \alpha = p - p^{-1} - p^{-3} - \&c. \\ \frac{1}{\alpha} = \quad p^{-1} + p^{-3} + \&c. \end{array}\right\}$$

$$\text{suppose } n = 2,\ \left.\begin{array}{l} \alpha^2 = p^2 - 2 - p^{-2} - \&c. \\ \frac{1}{\alpha^2} = \quad p^{-2} + \&c. \end{array}\right\}$$

&c. &c.

Hence it appears that when n is an integer, the first series will express the value of $2\cos.nx$ if we stop at the term in which the index of $p = 0$. See the Trig. App. p. 240.

The developements of the following example are taken from the Mecanique Celeste. In these developements u

and nt represent the anomalies of a planet's orbit, r the radius vector, and e the eccentricity; and since the eccentricity is small, the resulting series converge with great rapidity.

Ex. 13. First, let $nt = u - e \sin.u$; required u in terms of e.

$u = nt + e \sin.u$ $\therefore$ here

$$\left.\begin{array}{l} x = nt \\ y = e \\ fx = \sin.x \end{array}\right| \text{ hence, by the theorem Cor. 1. since } \phi x = x,$$

$$u = x + \frac{e}{1}\sin.x + \frac{e^2}{1.2}(\sin.^2 x)' + \frac{e^3}{1.2.3}(\sin.^3 x)'' + \&c.$$

But $(\sin.^2 x)' = \sin.2x$; $(\sin.^3 x)'' = 3\left\{2\sin.x\cos.^2 x - \sin^3 x\right\} = \frac{3}{4}\left\{8\sin.x - 9\sin.^3 x\right\} = \frac{1}{2^2}\left\{3^2\sin.3x - 3\sin.x\right\}$ (Trig. p. 47);

&c. = &c.; hence

$$u = nt + e\sin.nt + \frac{e^2}{1.2.2}2\sin.2nt + \frac{e^3}{1.2.3.2^2}\left\{3^2\sin.3nt - 3\sin.nt\right\}$$
$$+ \frac{e^4}{1.2.3.4.2^3}\left\{4^3\sin.4nt - 4.2^3\sin.2nt\right\} + \&c.$$

Next, let $r = a(1 - e\cos.u)$; required r in terms of e.

$r = \phi u = a(1 - e\cos.u)$ $\therefore$ $\phi x = a(1 - e\cos.x)$ $\therefore$ $\phi' x = ae\sin.x$ and $fx = \sin.x$; hence, by the theorem,

$$r = a(1 - e\cos.x) + ae^2\sin.^2 x + \frac{e^2}{1.2}(ae\sin.^3 x)' \dots$$
$$+ \frac{e^3}{1.2.3}(ae\sin.^4 x)'' + \&c.$$
$$= a(1 - e\cos.x) + \frac{ae^2}{2}(1 - \cos.2x) + \frac{ae^3}{1.2.2^2}(3\cos.x$$
$$- 3\cos.3x) \text{ (Trig. p. 45.)} + \&c.$$
$$\therefore \frac{r}{a} = 1 + \frac{e^2}{2} - e\cos.nt - \frac{e^2}{2}\cos.2nt - \frac{e^3}{1.2.2^2}(3\cos.3nt \dots$$
$$- 3\cos.nt) + \frac{e^4}{1.2.3.3^3}(4^2\cos.4nt - 4.2^2\cos.2nt) - \&c.$$

PRAXIS.

1. $x = z - az^3 + bz^5 - \&c.$ $\therefore$ $z = x + ax^3 + (3a^2 - b)x^5 + \&c.$ Alg. p. 190.

2. $yz^n + a - z = 0 \ \therefore\ z^m = a^m + ma^{m+n-1}\dfrac{y}{1} \ldots\ldots$

$+ m.(m+2n-1)\,a^{m+2n-2}\dfrac{y^2}{1.2} + \&c.$

3. $y = 1 + px + \dfrac{p^2x^2}{1.2} + \dfrac{p^3x^3}{1.2.3} + \&c. \ldots\ldots$

$\therefore\ x = \dfrac{1}{p}\left\{y - (e-1) + (e-1)^2 - ((e-1)^3 + \dfrac{e}{2}(e-1)^2) + \&c.\right\}.$

4. $z = 2 - \dfrac{y^2}{z}\ \therefore\ \dfrac{1}{z^n} = \dfrac{1}{2^n} + \dfrac{ny^2}{2^{n+2}} + \dfrac{n.(n+3)}{1.2}\dfrac{y^4}{2^{n+4}} \ldots$

$+ \dfrac{n.(n+3)\,(n+5)}{1.2.3}\dfrac{y^6}{2^{n+6}} + \&c.$

5. $u = x + e\sin.u\ \therefore\ lu = lx + \dfrac{e}{1}\dfrac{\sin.x}{x} + \dfrac{e^2}{1.2}\left(\dfrac{\sin.^2x}{x}\right)' \ldots$

$+ \dfrac{e^3}{1.2.3}\left(\dfrac{\sin.^3x}{x}\right)'' + \&c.$

6. $z = x + fz\ \therefore\ z^{-n} = \dfrac{1}{x^n} - \dfrac{nfx}{x^{n+1}} - \dfrac{n}{1.2}\left(\dfrac{fx^2}{x^{n+1}}\right)' \ldots$

$- \dfrac{n}{1.2.3}\left(\dfrac{fx^3}{x^{n+1}}\right)'' - \&c.$

43. *Required to develope* $u = \dfrac{Fz}{x - z + fz}$ *in a series ascending by the powers of* z.

Let the required developement be
$u = A_0 + A_1z + A_2z^2 + \ldots\ldots + A_nz^n + \&c.$

By actual division $u = \dfrac{Fz}{x-z} - \dfrac{Fzfz}{(x-z)^2} + \dfrac{Fzfz^2}{(x-z)^3} - \&c.,$

therefore $A_n = t^n.\dfrac{Fz}{x-z} - t^n.\dfrac{Fzfz}{(x-z)^2} + t^n.\dfrac{Fzfz^2}{(x-z)^3} - \&c.,$

if the symbol $t^n.\dfrac{Fz}{x-z}$ be taken to represent the coefficient of that term in the developement of $\dfrac{Fz}{x-z}$, which contains z^n. (Alg. 346.)

Now, to find these coefficients, assume

$\mathrm{F}z = \mathrm{B}_0 + \mathrm{B}_1 z + \mathrm{B}_2 z^2 + \ldots + \mathrm{B}_n z^n + \&c.$, then since

$\frac{1}{x-z} = \frac{1}{x} + \frac{z}{x^2} + \frac{z^2}{x^3} + \ldots + \frac{z^n}{x^{n+1}} + \&c.$, we have

$$t^n.\frac{\mathrm{F}z}{x-z} = \frac{\mathrm{B}_0}{x^{n+1}} + \frac{\mathrm{B}_1}{x^n} + \frac{\mathrm{B}_2}{x^{n-1}} + \ldots + \frac{\mathrm{B}_n}{x}$$

$$= \frac{\mathrm{B}_0 + \mathrm{B}_1 x + \mathrm{B}_2 x^2 + \ldots + \mathrm{B}_n x^n}{x^{n+1}}$$

$= \frac{(\mathrm{F})x}{x^{n+1}}$, if we place F in brackets, to denote that the developement of the function is to be restricted to the negative powers of x.

Similarly it may be shown that $t^n.\frac{\mathrm{F}z fz}{x-z} = \frac{(\mathrm{F})x\,(f)x}{x^{n+1}}$, the brackets denoting the same as before; hence, differentiating both the function $\frac{\mathrm{F}z fz}{x-z}$ and its developement partially with respect to x, we shall have

$$-t^n.\frac{\mathrm{F}z fz}{(x-z)^2} = \left(\frac{(\mathrm{F})x\,(f)x}{x^{n+1}}\right)'.$$

Repeating the process, since it can be shown that $t^n.\frac{\mathrm{F}z fz^2}{x-z} = \frac{(\mathrm{F})x\,(f)x^2}{x^{n+1}}$, there will result, after two differentiations,

$$t^n.\frac{\mathrm{F}z fz^2}{(x-z)^3} = \frac{1}{2}\left(\frac{(\mathrm{F})x\,(f)x^2}{x^{n+1}}\right)''; \text{ and so on.}$$

Hence $\mathrm{A}_n = \frac{(\mathrm{F})x}{x^{n+1}} + \left(\frac{(\mathrm{F})x\,(f)x}{x^{n+1}}\right)' + \frac{1}{1.2}\left(\frac{(\mathrm{F})x\,(f)x^2}{x^{n+1}}\right)'' + \&c.$

which, being the coefficient of the terminus generalis, gives the law of the required series. "Resolution des Equations Numeriques, Note 11."

Cor. By successively differentiating the original function partially with respect to x, we have

$$-\ u' = \frac{\mathrm{F}z}{(x-z+fz)^2}$$
$$+\ \frac{u''}{1.2} = \frac{\mathrm{F}z}{(x-z+fz)^3}$$
$$-\frac{u'''}{1.2.3} = \frac{\mathrm{F}z}{(x-z+fz)^4}$$
&c. = &c.

hence, $t^n . \frac{\mathrm{F}z}{(x-z+fz)^m} = \ldots\ldots \pm t^n . \frac{u^{\prime(m-1)}}{1.2\ldots(m-1)}$ or $= \frac{1}{1.2\ldots(m-1)} (\mathrm{A}_n)^{\prime(m-1)}$, and consequently the developement of $\frac{\mathrm{F}z}{(x-z+fz)^m}$ may be obtained from that of $\frac{\mathrm{F}z}{x-z+fz}$.

Ex. 1. $u = \frac{\mathrm{P}+\mathrm{Q}z}{1-pz+z^2}$, where $p = 2\cos.\theta$

$$u = \frac{\frac{\mathrm{P}}{p} + \frac{\mathrm{Q}}{p}z}{\frac{1}{p} - z + \frac{z^2}{p}} \quad \therefore \text{ here}$$

$\therefore$

$$x = \frac{1}{p}$$
$$\mathrm{F}x = \frac{\mathrm{P}}{p} + \frac{\mathrm{Q}x}{p}$$
$$fx = \frac{x^2}{p}$$

$$\frac{\mathrm{F}x}{x^{n+1}} = \frac{\mathrm{P}}{px^{n+1}} + \frac{\mathrm{Q}}{px^n}$$
$$\left(\frac{\mathrm{F}x\,fx}{x^{n+1}}\right)' = -\frac{n-1}{p^2}\frac{\mathrm{P}}{x^n} - \frac{n-2}{p^2}\frac{\mathrm{Q}}{x^{n-1}}$$

$$\left(\frac{\mathrm{F}x\,fx^2}{x^{n+1}}\right)'' = \frac{(n-2)(n-3)}{p^3}\cdot\frac{\mathrm{P}}{x^{n-1}} + \frac{(n-3)(n-4)}{p^3}\cdot\frac{\mathrm{Q}}{x^{n-2}}$$

&c. = &c.; hence

$$\mathrm{A}_n = \mathrm{P}\left\{p^n - (n-1)p^{n-2} + \frac{(n-2)(n-3)}{1.2}p^{n-4} \ldots\ldots - \frac{(n-3)(n-4)(n-5)}{1.2.3}p^{n-6} + \&c.\right\}$$
$$+\mathrm{Q}\left\{p^{n-1} - (n-2)p^{n-3} + \frac{(n-3)(n-4)}{1.2}p^{n-5} \ldots\ldots - \frac{(n-4)(n-5)(n-6)}{1.2.3}p^{n-7} + \&c.\right\}$$

which series are not to contain negative powers of p.

Hence $u = \text{P} + (\text{P}p + \text{Q})z + (\text{P}p^2 - \text{P} + \text{Q}p)\frac{z^2}{1.2} + \&c.$ which is the same series as that produced by actual division.

Cor. $\text{A}_n = \text{P}\frac{\sin.(n+1)\theta}{\sin.\theta} + \text{Q}\frac{\sin.n\theta}{\sin.\theta}$ (vid. 2. 39 (δ)).

Ex. 2. Required to develope $\frac{\text{P}+\text{Q}z}{(1-pz+z^2)^2}$.

By the preceding example, if $u = \frac{\text{P}+\text{Q}z}{1-pz+z^2}$, $t^n.u \ldots$

$$= \frac{\text{P}}{p}\left\{\frac{1}{x^{n+1}} - \frac{n-1}{p}\frac{1}{x^n} + \frac{(n-2)(n-3)}{1.2\,p^2}\frac{1}{x^{n-1}} - \&c.\right\}$$
$$+ \frac{\text{Q}}{p}\left\{\frac{1}{x^n} - \frac{n-2}{p}\frac{1}{x^{n-1}} + \frac{(n-3)(n-4)}{1.2\,p^2}\frac{1}{x^{n-2}} - \&c.\right\}$$

therefore $t^n.\frac{\text{P}+\text{Q}z}{(1-pz+z^2)^2}$, which $= -t^n.\frac{u'}{1}$,

$$= \frac{\text{P}}{p}\left\{\frac{n+1}{x^{n+2}} - \frac{n(n-1)}{p}\frac{1}{x^{n+1}} + \frac{(n-1)(n-2)(n-3)}{1.2\,p^2}\frac{1}{x^n} - \&c.\right\}$$
$$+ \frac{\text{Q}}{p}\left\{\frac{n}{x^{n+1}} - \frac{(n-1)(n-2)}{p}\frac{1}{x^n} \ldots\ldots\ldots\ldots\right.$$
$$\left. + \frac{(n-2)(n-3)(n-4)}{1.2\,p^2}\frac{1}{x^{n-1}} - \&c.\right\}$$

$$= \text{P}\left\{(n+1)p^{n+1} - n.(n-1)p^{n-1} \ldots\ldots\ldots\ldots\right.$$
$$\left. + \frac{(n-1)(n-2)(n-3)}{1.2}p^{n-3} - \&c.\right\}$$
$$+ \text{Q}\left\{np^n - (n-1)(n-2)p^{n-2} \ldots\ldots\ldots\ldots\right.$$
$$\left. + \frac{(n-2)(n-3)(n-4)}{1.2}p^{n-4} - \&c.\right\};$$

the series being restricted as before to the positive powers of p.

44. *The sum of the reciprocals of the nth powers of the roots of the equation $x - z + fz = 0$ is equal to the sum of all the terms of the developement of z^{-n}, which contain the negative powers of x.*

For, let $x - z + fz = 0 = (z + a_1)(z + a_2) \ldots (z + a_n)$;

then, since $l(x-z+fz)=l(z+a_1)+l(z+a_2)+\ldots+l(z+a_n)$,

therefore $\dfrac{1-f'z}{x-z+fz}=\dfrac{1}{z+a_1}+\dfrac{1}{z+a_2}+\ldots+\dfrac{1}{z+a_n}$.

Substitute $\mathrm{F}z=1-f'z$; then, by actual division on both sides of the equation, if $\int_{-n}$ represent the sum of the reciprocals of the nth powers of the roots of the equation, we shall have (Alg. 346.)

$$\int_{-(n+1)}=t^n.\frac{\mathrm{F}z}{x-z+fz}=(\text{Art. 42.})\frac{(\mathrm{F})x}{x^{n+1}}+\left(\frac{(\mathrm{F})x(f)x}{x^{n+1}}\right)'+\ \&\text{c.}$$

Let $p=(f)x$, $q=\dfrac{1}{x^{n+1}}$; then $(\mathrm{F})x$, which $=1-(f')x,=1-p'$,

$$\&\int_{-(n+1)}=q(1-p')+(pq(1-p'))'+\frac{1}{1.2}(p^2q(1-p'))''\ \ldots$$
$$+\frac{1}{1.2.3}(p^3q(1-p'))'''+\&\text{c.}$$
$$=q-qp'+(pq)'-(pqp')'+\frac{1}{1.2}\left\{(p^2q)''-(p^2qp')''\right\}$$
$$+\frac{1}{1.2.3}\left\{(p^3q)'''-(p^3qp')'''\right\}+\ \&\text{c.}$$

But $(pq)'=\ qp'\ +\ pq'$ or $-qp'+(pq)'=pq'$

$\frac{1}{2}(p^2q)''=(pqp')'+\frac{1}{2}(p^2q')'$ or $-(pqp')'+\dfrac{1}{1.2}(p^2q)''\ldots$

$=\frac{1}{2}(p^2q')'$

$\frac{1}{3}(p^3q)'''=(p^2qp')''+\frac{1}{3}(p^3q')''$ or $-\dfrac{(p^2qp')''}{1.2}+\dfrac{(p^3q)'''}{1.2.3}$

$=\dfrac{1}{1.2}(p^3q')''$

&c. = &c.;

hence, by substitution,

$$\int_{-(n+1)}=q+pq'+\frac{1}{1.2}(p^2q')'+\frac{1}{1.2.3}(p^3q')''+\&\text{c.}$$
$$=\frac{1}{x^{n+1}}-\frac{n+1}{1}\frac{(f)x}{x^{n+2}}-\frac{n+1}{1.2}\left(\frac{(f)x^2}{x^{n+2}}\right)'-\ \&\text{c.}$$

In this equation, change $n+1$ into n and there results

$$\int_{-n}=\frac{1}{x^n}-\frac{n(f)x}{x^{n+1}}-\frac{n}{1.2}\left(\frac{(f)x^2}{x^{n+1}}\right)'-\frac{n}{1.2.3}\left(\frac{(f)x^3}{x^{n+1}}\right)''-\ \&\text{c.}$$

which = the sum of all the terms in the developement of z^{-n} that contain negative powers of x (Art. 41. Praxis, Ex. 6).

Ex. 1. Required to demonstrate the formula of Art. 41. Ex. 12.

The roots of $z^2-2\cos.x.\ z+1=0$, are of the form a, $\frac{1}{a}$ (Alg. 325); hence we may assume $a+\frac{1}{a}=2\cos.x=p$, and consequently (2. 37) $2\cos.nx=a^n+\frac{1}{a^n}=f_n=$, by the article, part of the developement of z^{-n}.

This has been developed in Art. 42, Ex. 6, if we make $a=1$, $c=1$, $b=p$ and $n=-n$; hence by that example

$$z^{-n}=p^n\left\{1-\frac{n}{p^2}+\frac{n.(n-3)}{1.2\,p^4}-\frac{n.(n-4)\,(n-5)}{1.2.3\,p^6}+\text{\&c.}\right\}$$

and $2\cos.nx=p^n-np^{n-2}+\frac{n.(n-3)}{1.2}p^{n-4}-\text{\&c.}$

In this example $f_n=f_{-n}$.

Ex. 2. Given $z^n-p_1z^{n-1}+p_2z^{n-2}-\ldots\pm p_n=0$; required f_n.

The transformed equation is $1-p_1z+p_2z^2-\ldots\pm p^nz^n=0$,

or $z=\frac{1}{p_1}+\frac{1}{p_1}\left\{p_2z^2-p_3z^3+\ldots\pm p_nz^n\right\}$; here

$$x=\frac{1}{p_1}$$

$$fx=\frac{1}{p_1}\left\{p_2x^2-p_3x^3+\ldots\pm p_nx^n\right\},$$ and by the article,

$$f_{-n}=(z^{-n})$$

$$=\frac{1}{x^n}-\frac{n(f)x}{x^{n+1}}-\frac{n}{1.2}\left(\frac{(f)x^2}{x^{n+1}}\right)'-\frac{n}{1.2.3}\left(\frac{(f)x^3}{x^{n+1}}\right)''-\text{\&c.}$$

But $\frac{1}{x^n}=p_1^{\ n}$

$$-\frac{n(f)x}{x^{n+1}}=-\frac{n}{p_1}\left\{\frac{p_2}{x^{n-1}}-\frac{p_3}{x^{n-2}}+\ldots\pm\frac{p_n}{x}\right\}$$

$$=-np_2p_1^{\ n-2}+np_3p_1^{\ n-3}-np_4p_1^{\ n-4}\ \ldots\ldots$$
$$+np_5p_1^{\ n-5}-\text{\&c.}$$

$$-\frac{n}{1.2}\left(\frac{(f)x^2}{x^{n+1}}\right)'=-\frac{n}{1.2\,p_1^{\ 2}}\left\{\frac{(p_2-p_3x+\ldots\pm p_nx^{n-2})^2}{x^{n-3}}\right\}'$$

$$= -\frac{n}{1.2\,p_1^2}\left\{\frac{p_2^2}{x^{n-3}} - \frac{2\,p_2 p_3}{x^{n-4}} + \&c.\right\}'$$

$$= \frac{n}{1.2\,p_1^2}\left\{(n-3)p_2^2 x^{-n+2} - 2(n-4)p_2 p_3 x^{-n+3} + \&c.\right\}$$

$$= \frac{n.(n-3)}{1.2}\,p_2^2 p_1^{n-4} - n(n-4)\,p_2 p_3 p_1^{n-5} + \&c.$$

&c. = &c.; hence

$$f_n = p_1^n - np_2 p_1^{n-2} + np_3 p_1^{n-3} \begin{matrix} - & np_4 \\ + & \frac{n(n-3)}{1.2}p_2^2 \end{matrix}\Bigg\} p_1^{n-4} + \begin{matrix} np_5 \\ -n(n-4)p_2 p_3 \end{matrix}\Bigg\} p_1^{n-5} - \&c.$$

This is Waring's series, who in the Meditationes Algebraicæ, Ch. 1. has calculated it as far as p_1^{n-8}.

The examples of this chapter are only a few of the numerous instances in which the principle of successive differentiation is applicable to the developement of functions. The subject itself is of the greatest utility in the mixed mathematics, and it has been handled by almost all scientific writers of any note from the days of Sir I. Newton, who was the first to see its importance. It would be inconsistent with the plan of this work to give a fuller account of the developement of functions into series, or even to enter into further details concerning Lagrange's theorem; for information on the latter subject I must refer the reader to the work from which this and the preceding article have been taken,—" Resolution des Equations Numeriques, Note 11."

Before we conclude this subject with the cases in which these developements fail, we shall subjoin two theorems which will be required in the second volume, the one due to Euler and the other to John Bernoulli.

45. *Let $u = fx$ be such a function of x that when $x = a$, $u = b$; then shall* $u = b + (x-a)\frac{p}{1} - (x-a)^2\frac{q}{1.2} + (x-a)^3\frac{r}{1.2.3} - $ *&c., where p, q, r, &c. are the fluxional coefficients of $u = fx$.*

For, since $u = fx$, therefore (ex. hyp.) $b = fa$, and by

Taylor's theorem $f(x+h) = u + p.\frac{h}{1} + q.\frac{h^2}{1.2} + r.\frac{h^3}{1.2.3}$ + &c. where h is indeterminate.

Suppose $a - x = h$, or $a = x + h$; hence, by substitution, fa, which $= b$, $= u - p.\frac{x-a}{1} + q.\frac{(x-a)^2}{1.2} - \frac{r(x-a)^3}{1.2.3} +$ &c.

and $\quad u = b + (x-a)\frac{p}{1} - (x-a)^2\frac{q}{1.2} + \frac{r(x-a)^3}{1.2.3} -$ &c.

(Euler Calc. Int. Vol. 1. C. 7.)

46. $\int y dx = yx - \frac{x^2}{1.2}\frac{dy}{dx} + \frac{x^3}{1.2.3}\frac{d^2y}{dx^2} -$ &c.

For $\int y dx = yx - \int x dy$ (2. 52)

$$= yx - \int . x dx \times \frac{dy}{dx}$$

$$= yx - \frac{x^2}{2}\frac{dy}{dx} + \int \frac{x^2 d^2y}{1.2\, dx}$$

$$= yx - \frac{x^2}{2}\frac{dy}{dx} + \int \frac{x^2 dx}{1.2} \times \frac{d^2y}{dx^2}$$

$$= yx - \frac{x^2}{1.2}\frac{dy}{dx} + \frac{x^3}{1.2.3}\frac{d^2y}{dx^2} - \int . \frac{x^3}{1.2.3} \times \frac{d^3y}{dx^2}$$

$=$ &c.

$$= yx - \frac{x^2}{1.2}\frac{dy}{dx} + \frac{x^3}{1.2.3}\frac{d^2y}{dx^2} - \frac{x^4}{1.2.3.4}\frac{d^3y}{dx^3} + \text{\&c.}$$

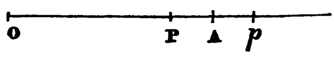
It is remarked of this theorem, that it has the same relation to the integral calculus which that of Taylor has to the fluxional.

CHAPTER V.

The cases in which the developement of functions fails.—Fractions whose numerator and denominator vanish at the same time.

1. It frequently happens in algebraick calculations, that by assigning a particular value to the variable the result becomes vain and nugatory. Thus, $\frac{x^2-a^2}{x-a}$, when $x = a$, becomes $\frac{0}{0}$; from which we are not to infer that the value of the function equals nothing, the only strict inference being, that in this particular case the algebraick rule of division ceases to be applicable; and the reason is obvious, that in the operation we suppose $x^2 - a^2$ and $x-a$ to represent real quantity, so that this particular value of x is excluded by the supposition. In such cases the rule, though the phrase is certainly incorrect, is said to *fail.*

It is upon the same principle that we explain, what at first sight appears paradoxical and inconsistent with the definition Ch. 1. Art. 7., that the fluxion of a quantity which vanishes is not necessarily equal to nothing.



Let OP $= x$, OA $= a$; then if P describe the line OAP uniformly, the fluxion of Ap or $a - x$ must be a constant quantity; a negative quantity before P coincides with A, and afterwards positive. It does not vanish, then, when P is at A, though AP $= 0$.

To take another example;

let $u = lx$, therefore $\frac{du}{dx} = \frac{1}{x}$; make $x = 1$, then $u = 0$; but $\frac{du}{dx} = 1$.

Or, let $u=(a-x)^2$ therefore $\frac{du}{dx}=2.(a-x)$, and $\frac{d^2u}{dx^2}=-2$; make $x=a$, then $u=0$, $\frac{du}{dx}=0$; but $\frac{d^2u}{dx^2}=-2$.

The student should attend to the following distinction; if $u=\mathrm{F}(x,y)=0$, x and y remaining indeterminate, $\frac{du}{dx}$ shall also $=0$; but if $y=fx$, and $y=0$, in consequence of assigning to x a particular value, it does not necessarily follow that $dy=0$.

The theorems of the preceding chapters never fail but in consequence of assigning a particular value to the variable.

Let $f(x+a)=(x+a)^{\frac{1}{2}}$, then by the theorem (Ch. 3. Art. 1.)

$$(x+a+h)^{\frac{1}{2}}=(x+a)^{\frac{1}{2}}+\frac{h}{(x+a+h)^{\frac{1}{2}}+(x+a)^{\frac{1}{2}}}$$

$$=(x+a)^{\frac{1}{2}}+\frac{h}{2.(x+a)^{\frac{1}{2}}}\ldots\ldots\ldots\ldots$$

$$-\frac{h^2}{2(x+a)^{\frac{1}{2}}\left\{(x+a+h)^{\frac{1}{2}}+(x+a)^{\frac{1}{2}}\right\}^2}$$

$$=(x+a)^{\frac{1}{2}}+\frac{h}{2(x+a)^{\frac{1}{2}}}-\frac{h^2}{8.(x+a)^{\frac{3}{2}}}+\&c.$$

either of which values may be substituted *generally* for $(x+a+h)^{\frac{1}{2}}$ in all calculations without fear of error; but if we suppose x to equal $-a$, the equation becomes $h^{\frac{1}{2}}=\infty$; and one cause of failure in this particular case is, that the value $x=-a$ is excluded by the supposition tacitly made in the theorem, that both $f(x+a)$ and $f(x+a+h)$ represent real quantities; but this is not the only cause, as will be shown in the following article.

2. *To explain the cases in which the theorems for the developement of functions fail.*

The theorem of Ch. 3. Art. 1. is found to fail only when a particular value is assigned to the variable.

In that part of the demonstration in which it is shown that there cannot be a term of the form $\mathrm{B}\sqrt[n]{h^m}$, it is assumed

that $f(x+h)$ must contain the same number of radicals as fx, which is true as a general proposition, but may cease to be true by assigning a particular value to the variable: ex. gr.

Let $fx = cx^2 + (x+b) \times \pm \sqrt{x+a}$; then $f(x+h) = c.(x+h)^2 + (x+h+b) \times \pm \sqrt{x+h+a}$, which contains the same number of radicals as fx, but suppose x equal to $-a$, then $fx = cx^2$, and $f(x+h) = c(h-a)^2 + (b+h-a) \times \pm h^{\frac{1}{2}}$, or $f(x+h)$ contains one radical more than fx, and consequently has a greater number of values, and the ex absurdo demonstration fails in this case.

The theorem then must be understood as applying only to functions of *indeterminate* variables.

If we take the fluxional coefficients of the above example we shall find that they each possess the same number of values; but if we suppose $x = -a$, then $fx = cx^2$, $f'x = \infty$, $f''x = \infty$, &c. or the developement fails.

It appears then, that if the developement fails, a value must have been assigned to the variable, which causes one or more radicals to disappear in the function. But the converse proposition does not hold good, that if a radical is made to disappear in the function, the developement necessarily fails. It will not fail in those cases in which the radical disappears in consequence of its *coefficient* being made to vanish; thus, if in the same example we suppose x to equal $-b$, the radical disappears in fx, but the fluxional coefficients do not fail, for they become $f'x = -2bc + \sqrt{a-b}$,

$$f''x = 2c + \frac{1}{2\sqrt{a-b}}, \text{ \&c.} = \text{\&c.}$$

If we take the example $fx = cx^2 + (x+b)^2 \times \sqrt{x+a}$, and suppose as before $x = -b$, the radical disappears in $f'x$, but reappears in $f''x$, $f'''x$, &c.; and generally if we have $fx = cx^2 + (x+b)^m \sqrt{x+a}$, the radical disappears in the first $m-1$ fluxional coefficients, and reappears in all that follow; so that in all these cases the function can be developed by means of the theorem.

There is this difference between Taylor's theorem and Maclaurin's: the first cannot fail so long as the variable is indeterminate; but since the fluxional coefficients in the latter are obtained by assigning to the variable a particular value, it may fail from the nature of the function which is

to be developed. Thus the fluxional coefficients of $\frac{1}{x}$ and of lx, deduced from Maclaurin's theorem, are all infinite. Vid. Ch. 4. 6. Ex. 3. In these two instances the functions are not developable in series ascending by the powers of x. There are cases in which, by adopting certain artifices of calculation, the failure of Maclaurin's theorem may be prevented. Vid. Ch. 4. 9.

It may further be observed, that Taylor's theorem always fails when the assigned value of x causes any of the terms to become imaginary, and that this may take place without causing the function itself to be imaginary; thus take $fx = c + x^2\sqrt{x-a}$; if we suppose $x = 0$, $fx = c$, $f'x = 0$; but $f''x$, $f'''x$.... all contain $\sqrt{-1}$.

3. When one of the terms of the developement becomes infinite and it fails, in consequence of supposing $x = a$, we must substitute $x + h$ for x, h being *indeterminate*, and a series will arise which will not fail when $x = a$. In this case some of the terms will contain fractional powers of p.

Thus, let $fx = 2ax - x^2 + a\sqrt{x^2 - a^2}$; then, by Taylor's theorem, we have $f(x+h) = 2ax - x^2 + a\sqrt{x^2 - a^2} + \left\{2(a-x) + \frac{ax}{\sqrt{x^2 - a^2}}\right\}\frac{h}{1} +$ &c., the second term of which fails when $x = a$; but substitute $x + h$ for x in the original equation, and there results

$f(x+h) = 2ax + 2ah - x^2 - 2xh + h^2 - a\sqrt{x^2 + 2xh + h^2 - a^2}$ + &c., in which, if we suppose $x = a$, we have

$f(a+h) = a^2 - h^2 + a\sqrt{2ah + h^2} + \text{\&c.} = a^2 - h^2 + a\text{A}h^{\frac{1}{2}} + a\text{B}h^{\frac{3}{2}}$ + &c., where A, B, &c. are the coefficients obtained by the binomial theorem.

4. *If $f(x+h)$, when a particular value is assigned to x, is developed in a series containing a radical of h, the exponent of which lies between n and $n+1$, all the fluxional coefficients to the nth order inclusive can be had from Taylor's theorem, but all the succeeding coefficients, as given by that theorem, are infinite.*

Let $f(x+h)$ contain a radical of the form $(x-a)^{\frac{1}{z}}$, and when $x=a$ let one of the terms of the developement contain a radical of h, viz. $h^{n+\frac{1}{z}}$ where $\frac{1}{z}$ is a proper fraction.

Suppose $f(a+h)$ when developed to equal $A+Bh+Ch^2+\ldots\ldots+Mh^n+Nh^{n+\frac{1}{z}}+$&c.; and to find the coefficients, suppose $h=0$, then $A=fa$.

Next, since (Ch. 4. 3.) $\frac{df(a+h)}{da}=\frac{d.f(a+h)}{dh}$, $\frac{d^2f(a+h)}{da^2}=\frac{d^2f(a+h)}{dh^2}$, &c. = &c.; therefore we have

$$\frac{df(a+h)}{da}=B+2Ch+\ldots+nMh^{n-1}\ldots\ldots\ldots\ldots+\left(n+\frac{1}{z}\right)Nh^{n+\frac{1}{z}-1}+\text{\&c.}$$

$$\frac{d^2f(a+h)}{da^2}=2C+\ldots+n.\overline{n-1}\,Mh^{n-2}\ldots\ldots\ldots\ldots+\left(n+\frac{1}{z}\right)\left(n+\frac{1}{z}-1\right)Nh^{n+\frac{1}{z}-2}+\text{\&c.}$$

&c. = &c.

$$\frac{d^{n-1}f(a+h)}{da^{n-1}}=n.\overline{n-1}\ldots 2.1\,Mh\ldots\ldots\ldots\ldots+\left(n+\frac{1}{z}\right)\ldots\left(\frac{1}{z}+3\right)\left(\frac{1}{2}+2\right).\,Nh^{\frac{1}{z}+1}+\text{\&c.}$$

$$\frac{d^nf(a+h)}{da^n}=Q.Nh^{\frac{1}{z}}+\text{\&c.}$$

Hence, making $h=0$; $B=\frac{dfa}{da}$, $C=\frac{d^2fa}{da^2}$, ... $M=\frac{d^{n-1}fa}{1.2\ldots nda^{n-1}}$, $N=\frac{d^nfa}{Qda^n}$; which are their values as deduced from Taylor's theorem.

But differentiating again, $\frac{d^{n+1}f(a+h)}{da^{n+1}}=\frac{1}{z}\,QNh^{\frac{1}{z}-1}+$&c. $=\frac{1}{z}\frac{QN}{h^{1-\frac{1}{z}}}$ for $\frac{1}{z}$ is a proper fraction, $=\infty$ when $h=0$.

Similarly it may be shown that $\frac{d^{n+2}fa}{da^{n+2}}$, &c. &c. all become infinite when $x=a$.

5. *Fractions which take the form of* $\frac{0}{0}$.

When a fraction takes the form of $\frac{0}{0}$ in consequence of assigning a particular value to its variable, there are different methods by which its real value may be ascertained.

One method is to put it under a different form, and then substitute the particular value of x; thus $\frac{x^2-a^2}{x-a}=x+a=2a$ when $x=a$.

Another is founded upon the principles of prime and ultimate ratios; if we suppose x to increase from the value of $x=a$, the nascent ratio of x^2-a^2: $x-a$ is the required value of $\frac{x^2-a^2}{x-a}$; and to obtain this, suppose $x=a+h$, then the fraction becomes $\frac{2ah+h^2}{h}$ or $2a+h$, which ultimately $=2a$. This method is in general to be applied when the fraction contains radicals.

Upon the same principle as the last is the method of differentiating both the numerator and denominator. Taking the same example, we have $\frac{d.\text{num}^r}{d.\text{den}^r}=\frac{2xdx}{dx}=2x=2a$ when $x=a$. Again, to find $\frac{x^n-1}{x-1}$, when $x=1$, we have $\frac{d.\text{num}^r}{d.\text{den}^r}=nx^{n-1}=n$ when $x=1$.

In adopting this method, we sometimes obtain the same result $\frac{0}{0}$; thus to find $\frac{ax^2-2acx+ac^2}{bx^2-2bcx+bc^2}$, when $x=c$; $\frac{d.\text{num}^r}{d.\text{den}^r}=\frac{2ax-2ac}{2bx-2bc}=\frac{0}{0}$ when $x=c$. But differentiate a second time, and we have $\frac{d^2.\text{num}^r}{d^2.\text{den}^r}=\frac{2a}{2b}=\frac{a}{b}$, which is the value of the required fraction when $x=c$.

6. *To explain the principle upon which the value of fractions whose numerator and denominator vanish at the same time is obtained by successive differentiations.*

Let the fraction be $\frac{N}{D}$; then if N and D both vanish when

a particular value a is assigned to the variable, they must have a common divisor of the form $x - a$.

First, suppose $\frac{N}{D} = \frac{P.(x-a)}{Q.(x-a)}$, where P and Q do not contain $x - a$; then, differentiating, $\frac{dN}{dD} = \frac{dP.(x-a)+P}{dQ.(x-a)+Q} = \frac{P}{Q}$, which do not vanish when we substitute a for x, and thus the required value of the fraction is obtained.

Next, suppose the common factor to be $(x-a)^2$, or $\frac{N}{D} = \frac{P.(x-a)^2}{Q.(x-a)^2}$; here $\frac{dN}{dD} = \frac{dP.(x-a)^2 + 2P.(x-a)}{dQ.(x-a)^2 + 2Q.(x-a)}$, which $= \frac{0}{0}$ when a is substituted for x; but differentiating a second time, we have $\frac{d^2N}{d^2D} = \frac{d^2P.(x-a)^2 + 4dP.(x-a) + 2P}{d^2Q.(x-a)^2 + 4dQ(x-a) + 2Q} = \frac{P}{Q}$, the required value.

Generally, if $\frac{N}{D} = \frac{P.(x-a)^m}{Q.(x-a)^m}$, where m is a positive integer, by differentiating N and D m times, and substituting a for x in the result, we shall obtain $\frac{N}{D} = \frac{P}{Q}$.

7. If $\frac{N}{D} = \frac{P.(x-a)^m}{Q.(x-a)^n} = \frac{P}{Q}.(x-a)^{m-n}$, and we differentiate n times, and substitute a for x in the result, we shall have $\frac{N}{D} = \frac{0}{Q} = 0$, m being greater than n; but if m be less than n, differentiating m times we shall have $\frac{N}{D} = \infty$. Hence the Rule, that "*If in the foregoing process the numerator vanishes when the denominator is finite, the fraction is* $= 0$; *but if, on the contrary, the numerator is finite and the denominator vanishes, the fraction* $= \infty$.

8. *This method is not applicable when the common factor of the numerator and the denominator is a radical.*

For at each differentiation the index of the factor is dimi-

nished by unity; if then it is a proper fraction, it will become negative without passing through zero, and the fraction will take the form of $\frac{\infty}{\infty}$.

The circumstance of the common factor being a radical will show itself by the numerator and the denominator becoming infinite at the same time; for when the index of the factor becomes negative, or of the form $(x-a)^{-\frac{1}{n}}$, N and D each $=\infty$, when $x=a$. In such cases we must proceed by the method of limits. And, in general, this is the method which should be adopted whenever the function contains fractional indices.

9. The preceding articles are applicable only to *algebraick* functions, but it may be shown likewise of transcendentals, that when they take the form of $\frac{0}{0}$ they have a determinate value which may be found by successive differentiations.

For, let u take the form of $\frac{0}{0}$ in consequence of x being $=a$, then $u=\frac{F(x-a)}{f(x-a)}=$, by substitution, $\frac{Fh}{fh}=$, by assumption, $\frac{Ah^{\alpha}+Bh^{\beta}+\&c.}{A'h^{\alpha'}+B'h^{\beta'}+\&c.}$, where the terms are arranged in an ascending order, and the coefficients are independent of h.

Now α is either greater than, equal to, or less than α'; if greater, $(u)=0$; if less, $(u)=\infty$; but if equal, $(u)=\frac{A+Bh^{\beta-\alpha}+\&c.}{A'+B'h^{\beta'-\alpha'}+\&c.}=\frac{A}{A'}$, a determinate value, which may be found if the numerator and denominator of u can be developed in series ascending by the powers of h. Hence it appears that, whether the functions are algebraick or transcendental, if we substitute $a+h$ for x, and expand the numerator and the denominator in series ascending by the powers of h, the first terms of the series will give the true value of the function.

If $A=0$ and $A'=0$, the series must be expanded as far as their second terms; and so on.

10. *Lagrange's demonstration of Arts.* 6 *and* 9.

Let $u = \frac{\text{F}x}{fx}$, $\text{F}x$ and fx both being $= 0$ when $x = a$; therefore $ufx = \text{F}x$, and, differentiating, $u'fx + uf'x = \text{F}'x$, but $fx = 0$ when $x = a$, therefore the equation becomes $(u)\,(f'x) = (\text{F}'x)$, or $(u) = \frac{(\text{F}'x)}{(f'x)}$.

If $(\text{F}'x)$ and $(f'x)$ also vanish, it follows from what has been already proved that $(u) = \frac{(\text{F}''x)}{(f''x)}$, &c. &c.

11. *The fractions* $\frac{(\text{F}x)}{(fx)}$, $\frac{(\text{F}'x)}{(f'x)}$, $\frac{(\text{F}''x)}{(f''x)}$, *&c. cannot become* $\frac{0}{0}$ *ad infinitum.*

For, if possible, let each of the functions $(\text{F}x)$, $(\text{F}'x)$, &c. (fx), $(f'x)$, &c. $= 0$; then, since $\text{F}(x+h) = \text{F}x + \frac{h}{1}\text{F}'x + \frac{h^2}{1.2}\text{F}''x +$ &c., therefore we have $\text{F}(a+h) = 0$ whatever be the value of h, which is absurd, or $x = a$ does not cause all the functions to vanish.

Each of these functions may become *infinite* ad inf., in which case the fraction is indeterminate; this happens (Vid. Art. 2.) when the developement of $f(x+h)$ fails in consequence of a radical disappearing in fx, which does not disappear in $f(x+h)$.

12. *When both numerator and denominator become* infinite *at the same time, the fraction may be made to take the form* $\frac{0}{0}$.

$$\text{For } \frac{\text{N}}{\text{D}} = \frac{\text{X}}{\text{X}'} = \frac{\frac{1}{\text{X}'}}{\frac{1}{\text{X}}} = \frac{0}{0}, \text{ if X and X}' \text{ are infinite.}$$

13. *A product composed of two factors, one of which becomes infinite and the other equal to zero, may be made to take the form of* $\frac{0}{0}$.

For $\mathrm{x}\,.\,\mathrm{x}' = \dfrac{\mathrm{x}}{\dfrac{1}{\mathrm{x}'}} = \dfrac{0}{0}$, if $\mathrm{x} = 0$ and $\mathrm{x}' = \infty$.

14. *The difference of two functions which become infinite in consequence of assigning a particular value to the variable may be made to take the form of* $\dfrac{0}{0}$.

For $\dfrac{1}{\mathrm{x}} - \dfrac{1}{\mathrm{x}'} = \dfrac{\mathrm{x}' - \mathrm{x}}{\mathrm{x}\mathrm{x}'}$ *i.e.* when $\mathrm{x} = 0$ and $\mathrm{x}' = 0$, $\infty - \infty = \dfrac{0}{0}$.

It appears, from Ch. 1. 38., that the difference of two infinite functions may be a finite and determinate magnitude.

Hence, to find the value of $\dfrac{1}{\mathrm{x}} - \dfrac{1}{\mathrm{x}'}$ when $\mathrm{x} = 0$ & $\mathrm{x}' = 0$, the rule is, *Reduce the fractions to a common denominator, and if the resulting fraction takes the form of* $\dfrac{0}{0}$, *find its value either by the method of limits or by successive differentiations.*

15. *Examples.*

Ex. 1. $\dfrac{\mathrm{N}}{\mathrm{D}} = \dfrac{(x^2 - a^2)^{\frac{3}{2}}}{(x - a)^{\frac{3}{2}}}$ when $x = a$.

By the method of differentiation, we have

$\dfrac{d^2\mathrm{N}}{d^2\mathrm{D}} = \dfrac{4x^2(x^2 - a^2)^{-\frac{1}{2}}}{(x - a)^{-\frac{1}{2}}} = \dfrac{\infty}{\infty}$ when $x = a$; assume therefore

$x = a + h \;\therefore\; \dfrac{\mathrm{N}}{\mathrm{D}} = \dfrac{(2ah + h^2)^{\frac{3}{2}}}{h^{\frac{3}{2}}} = 2a^{\frac{3}{2}}$, when $x = a$.

Ex. 2. $\dfrac{\mathrm{N}}{\mathrm{D}} = \dfrac{(x^2 - 3ax + 2a^2)^{\frac{2}{3}}}{(x^3 - a^3)^{\frac{1}{2}}}$ when $x = a$.

Assume $x = a + h \;\therefore\; \dfrac{\mathrm{N}}{\mathrm{D}} = \dfrac{(h^2 - ah)^{\frac{2}{3}}}{(3a^2h + 3ah^2 + h^3)^{\frac{1}{2}}} \ldots\ldots$

$= \dfrac{h^{\frac{1}{6}}(h - a)^{\frac{2}{3}}}{(3a^2 + 3ah + h^2)^{\frac{1}{2}}} = \dfrac{0}{(3a^2)^{\frac{1}{2}}}$ when $x = a$, or $\dfrac{\mathrm{N}}{\mathrm{D}} = 0$.

Ex. 3. $\dfrac{\text{N}}{\text{D}} = \dfrac{x^{\frac{1}{2}} - a^{\frac{1}{2}} + (x-a)^{\frac{1}{2}}}{(x^2 - a^2)^{\frac{1}{2}}}$ when $x = a$.

$$=\text{, by substitution, } \frac{(a+h)^{\frac{1}{2}} - a^{\frac{1}{2}} + h^{\frac{1}{2}}}{(2ah + h^2)^{\frac{1}{2}}} \quad . \; .$$

$$= \frac{\frac{1}{2}\, a^{-\frac{1}{2}} h + \&c. + h^{\frac{1}{2}}}{h^{\frac{1}{2}} (2a + h)^{\frac{1}{2}}} = \frac{\frac{1}{2} a^{-\frac{1}{2}} h^{\frac{1}{2}} + \&c. + 1}{(2a+h)^{\frac{1}{2}}}$$

$$=\text{, ultimately, } \frac{1}{(2a)^{\frac{1}{2}}}.$$

Ex. 4. $u = \dfrac{\text{N}}{\text{D}} = \dfrac{a^x - b^x}{x}$, when $x = 0$.

$$\frac{d\text{N}}{d\text{D}} = la.\, a^x - lb.\, b^x \;\therefore\; (u) = la - lb = l\frac{a}{b}.$$

Otherwise. $a^x = 1 + la.\, x + la^2.\dfrac{x^2}{1.2} + \&c.$

$$x = 1 + lb.\, x + lb^2.\frac{x^2}{1.2} + \&c.$$

$$\therefore \frac{a^x - b^x}{x} = la - lb + (la^2 - lb^2)\frac{x}{1.2} + \&c.$$

$$= la - lb = l\frac{a}{b} \text{ when } x = 0.$$

Ex. 5. $u = \dfrac{1 - \sin.x + \cos.x}{\sin.x + \cos.x - 1}$, when $x = 90^\circ = \dfrac{\pi}{2}$.

$$\frac{d\text{N}}{d\text{D}} = \frac{-\cos.x - \sin.x}{\cos.x - \sin.x} \;\therefore\; (u) = \frac{-1}{-1} = 1.$$

Ex. 6. $u = (1-x)\ \tan.\dfrac{\pi x}{2}$, when $x=1$.

$$\frac{\text{N}}{\text{D}} = \frac{1-x}{\cot.\frac{\pi x}{2}} \;\therefore\; \frac{d\text{N}}{d\text{D}} = \frac{-1}{-\operatorname{cosec.}^2\frac{\pi x}{2} \times \frac{\pi}{2}} \quad . \; . \; . \; .$$

$$\therefore (u) = \frac{-1}{-\frac{\pi}{2}} = \frac{2}{\pi}.$$

Ex. 7. $u = \dfrac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$.

Let $x = a - h$ $\therefore$ $2a^3x - x^4 = a^4 + 2a^3h + \&c.$ $\therefore$ $\sqrt{2a^3x - x^4}$ =, ultimately, $a^2 + ah$.

Also, $a\sqrt[3]{a^2x} = a\,{}^3\!/\overline{a^3 - a^2h}$ =, ultimately, $a.\left(a - \dfrac{h}{3}\right)$ $= a^2 - \dfrac{ah}{3}$ $\therefore$ N ultimately $= a^2 + ah - \left(a^2 - \dfrac{ah}{3}\right) = \dfrac{4ah}{3}$.

Again, ${}^4\!/\overline{ax^3} = \sqrt[4]{a^4 - 3a^3h + \&c.}$ =, ultimately, $a - \dfrac{3h}{4}$ $\therefore$

D ultimately $= a - \left(a - \dfrac{3h}{4}\right) = \dfrac{3h}{4}$ $\therefore$ $(u) = \dfrac{16a}{9}$.

Ex. 8. $u = \dfrac{a\sqrt[3]{4a^3 + 4x^3} - ax - a^2}{\sqrt{2a^2 + 2x^2} - a - x}$.

If in this example $a + h$ be substituted for x, and h^2, h^3, &c. be neglected, $u = \dfrac{0}{0}$; hence we must include the terms which contain h^2; or $4a^3 + 4x^3 = 8a^3 + 12a^2h + 12ah^2$; and to find an approximate cube root of this, let ${}^3\!\sqrt{8a^3 + 12a^2h + 12ah^2} = 2a + h + v$ where v contains h^2 $\therefore$ $8a^3 + 12a^2h + 12ah^2 = 8a^3 + 12a^2h + 12a^2v + 6ah^2$, where all the terms are neglected which contain higher powers than h^2, hence

$v = \dfrac{h^2}{2a}$ :: N$=\left(2a + h + \dfrac{h^2}{2a} - (2a + h)\right)a$.

Also, D$= 2a + h + \dfrac{h^2}{4a} - (2a + h)$ $\therefore$ $(u) = \dfrac{h^2}{2} \div \dfrac{h^2}{4a} = 2a$.

Ex. 9. $u = \dfrac{lx}{x^n}$ when $x = \infty$.

$x^n = 1 + \dfrac{nlx}{1} + \dfrac{n^2lx^2}{1.2} + \dfrac{n^3lx^3}{1.2.3} + \&c.$ $\therefore$ (Exp. Theor.)

$$u = \frac{lx}{1 + \dfrac{nlx}{1} + \dfrac{n^2lx^2}{1.2} + \&c.} = \frac{1}{\dfrac{1}{lx} + n + \dfrac{n^2lx}{1.2}} + \&c. \ldots$$

$\therefore$ $(u) = 0$, if n be finite.

Otherwise. By the method of differentiation

$$(u) = \frac{\frac{1}{x}}{nx^{n-1}} = \frac{1}{nx^n} = 0.$$

Ex. 10. $u = x^n \times lx$, when $x = 0$.

Substitute $y = \frac{1}{x}$ $\therefore u = \frac{l.\frac{1}{y}}{y^n}$ $\therefore (u) = \frac{-1}{ny^n} = 0.$

Cor. 1. $x^n \times lx^r = 0$.

Cor. 2. $0^0 = 1$.

For $h^h = 1 + \frac{hlh}{1} + \frac{h^2\,lh^2}{1.2} + \frac{h^3\,lh^3}{1.2.3} + \&c. = 1$, when $h = 0$.

Ex. 11. $u = \tan.x^{\cos.x}$ when $x = \frac{\pi}{2}$.

(4. 8. Ex. 4.) $u = x^{\cos.x} + \frac{2x^{3\cos.x}}{1.2.3} + \frac{16x^{5\cos.x}}{1.2.3.4.5} + \&c.$ $\therefore$

$$(u) = 1 + \frac{2}{1.2.3} + \frac{16}{1.2.3.4.5} + \&c.$$

Ex. 12. $u = \tan x^{\cos.x}$ when $x = \frac{\pi}{2}$.

$$u = \frac{\sin x^{\cos.x}}{\cos x^{\cos.x}} \quad \therefore (u) = \frac{1}{0^0} = 1 \text{ (Ex. 10. Cor. 2.)}$$

Otherwise. $\cos x^{\cos.x} = 1 + l\cos.x.\cos.x + l\cos.x^2.\frac{\cos.^2 x}{1.2}$ + &c.; but it may be shown that $(l\cos.x.\ \cos.x) = 0$, $(l\cos.x.^2\ \cos.^2 x) = 0$, &c. &c. $\therefore (u) = 1$.

Ex. 13. $u = \frac{x}{x-1} - \frac{1}{lx}$ when $x = 1$.

$(u) = \infty - \infty$, and if the fractions be reduced to a common denominator, according to Art. 14, we have $u = \frac{xlx - x + 1}{(x-1)lx}$, which $= \frac{0}{0}$ when $x = 1$, $=$, substituting $1 + h$ for x,

$$\frac{(1+h)\,l(1+h) - h}{h.l\,(1+h)} = \frac{\frac{h^2}{2} - \frac{h^3}{2.3} + \&c.}{h^2 - \frac{h^3}{2} + \&c.} \quad \therefore (u) = \tfrac{1}{2}.$$

Ex. 14. $u = \frac{1}{2x^2} - \frac{\pi}{2x \tan.\pi x}$ when $x = 0$.

$$u = \frac{\sin.\pi x - \pi x \cos.\pi x}{2x^2 \sin.\pi x} \therefore (u) = \frac{\pi^2 x \sin.\pi x}{4x \sin.\pi x + 2\pi x^2 \cos.\pi x} =$$

$$\frac{\pi^2 \sin.\pi x}{4 \sin.\pi x + 2\pi x \cos.\pi x} = \frac{\pi^3 \cos.\pi x}{6\pi \cos.\pi x - 2\pi^2 x \sin.\pi x} = \frac{\pi^2}{6}.$$

16. PRAXIS.

1. $u = \frac{a^4 - x^4}{a^2 - x^2}$, when $x = a$, $\therefore (u) = 2a^2$.

2. $u = \frac{x - x^n}{1 - x}$, when $x = 1$, $\therefore (u) = n - 1$.

3. $u = \frac{1 - (n+1)x^n + nx^{n+1}}{1 - 2x + x^2}$, when $x = 1$, $\therefore (u) = \frac{n.(n+1)}{2}$.

4. $u = \frac{x^3 - ax^2 - a^2x + a^3}{x^2 - a^2}$, when $x = a$, $\therefore (u) = 0$.

5. $u = \frac{ax - x^2}{a^4 - 2a^3x + 2ax^3 - x^4}$, when $x = a$, $\therefore (u) = \infty$.

6. $u = \frac{1 - x + lx}{1 - \sqrt{2x - x^2}}$, when $x = 1$, $\therefore (u) - 1$.

7. $u = (x - 1)\, l.\frac{1}{1 - x}$, when $x = 1$, $\therefore (u) = 0$.

8. $u = \frac{x^x - x}{1 - x + lx}$, when $x = 1$, $\therefore (u) = -2$.

9. $u = \frac{1}{lx} - \frac{x}{lx}$, when $x = 1$, $\therefore (u) = -1$.

10. $u = \frac{a - x - ala + alx}{a - \sqrt{2ax - x^2}}$, when $x = a$, $\therefore (u) = -1$.

11. $u = \frac{x^2 - a^2}{x^2}. \tan.\frac{\pi x}{2a}$, when $x = a$, $\therefore (u) = -\frac{4}{\pi}$.

12. $u = \frac{e^x - e^{\sin.x}}{x - \sin.x}$, when $x = 0$, $\therefore (u) = 1$.

13. $u = \frac{1}{1 - x} - \frac{2}{1 - x^2}$, when $x = 1$, $\therefore (u) = -\frac{1}{2}$.

14. $u = x \tan.x - \frac{\pi}{2} \sec.x$, when $x = \frac{\pi}{2}$, $\therefore (u) = -1$.

15. $u = \frac{\cot. \frac{a}{2x}}{2^x}$, when $x = \infty$, $\therefore (u) = \frac{1}{a}$.

16. $u = \dfrac{\tan.x - \sin.x}{\sin.x^3}$, when $x=0$, $\therefore (u) = \frac{1}{2}$.

17. $u = \left(1 + \dfrac{1}{x}\right)^x$, when $x = \infty$, $\therefore (u) = e$.

18. $u = \dfrac{a^{a-n} - x^{x-n}}{a-x}$, when $x=a$, $\therefore (u) = a^{a-n}.\left\{la + \dfrac{a-n}{a}\right\}$.

19. $u = \dfrac{\sqrt{2a^3 - x^4} - \sqrt[3]{ax^3}}{a - \sqrt[4]{ax^3}}$, when $x=a$, $\therefore (u) = 5a$.

20. $u = \dfrac{(2a^2 + 2x^2)^{\frac{1}{2}} - 2a^{\frac{2}{3}}x^{\frac{1}{3}}}{x-a}$, when $x=a$, $\therefore (u) = \frac{1}{3}$.

21. $u = \dfrac{a - \sqrt{2ax - x^2}}{a^{\frac{2}{3}} - \sqrt[3]{2ax - x^2}}$, when $x=a$, $\therefore (u) = \frac{3}{2}a^{\frac{1}{3}}$.

22. $u = \dfrac{x^3 - 4ax^2 + 7a^2x - 2a^3 - 2a^2\sqrt{2ax - a^2}}{x^2 - 2ax - a^2 + 2a\sqrt{2ax - x^2}}$, when $x=a$, $\therefore (u) = -5a$.

23. $u = \dfrac{a - x - a^2(a^2 + 8ax)^{-\frac{1}{2}}}{4ax + a^2 - 2x^2 - a\sqrt{a^2 + 8ax}}$, when $x = 0$, $\therefore (u) = -\infty$.

24. $u = \dfrac{\pi x - 1}{2x^2} + \dfrac{\pi}{x(e^{2\pi x} - 1)}$, when $x=0$, $\therefore (u) = \dfrac{\pi^2}{6}$.

25. $u = \dfrac{1}{2x^2} - \dfrac{\pi}{2x\tan.\pi x} - \dfrac{1}{n^2 - x^2}$, when $x = 0$, ...

$\therefore (u) = \dfrac{\pi^2}{6} - \dfrac{1}{n^2}$.

In the application of the rule contained in Art. 14, the student must be careful to ascertain that the resulting fraction takes the form of $\frac{0}{0}$. By successive differentiation, the factor which causes either the numerator or denominator to vanish will at length disappear, and a value may thus be attributed to the function which does not belong to it. Thus $\dfrac{1}{4x} \pm \dfrac{1}{2x(e^{\pi x} - 1)}$ when $x=0$, after two differentiations take each the form of $\frac{\pi}{8}$, but their true value is infinite.

17. *Required to investigate the value of fractions of the form* $\frac{0}{0}$ *which contain more than one variable.*

Let u contain two variables, x and y; and first suppose that $(u) = \frac{0}{0}$ in consequence of assigning a particular value to only one of the variables.

In this case, proceeding as in the above examples, (u) will always be found to have a *determinate* value.

Thus, if $u = \frac{c(x^2 - a^2)}{y(x-a) + (x-a)^2}$, $(u) = \frac{2ac}{y}$ when $x = a$.

Next, let $u = \frac{0}{0}$ in consequence of assigning particular values to both of the variables.

Here the value of (u) is indeterminate; for instance, let $u = \frac{c(x-a)}{y-b} = \frac{0}{0}$, when $x = a$ and $y = b$; but unless we know the ultimate ratio of $x - a : y - b$, it is obvious that the value of (u) cannot be determined.

18. There are instances, however, in which (u) may be determined without knowing this ratio; take for example $u = \frac{(x-y)a^n - (a-y)x^n + (a-x)y^n}{(x-y)(a-y)(a-x)}$, which $= \frac{0}{0}$ when $x = y = a$.

Substitute $x = a + h$ and $y = a + k$, then
$(u) = \frac{(h-k)a^n + k(a+h)^n - h(a+k)^n}{(h-k)hk}$ = (by expansion and reduction) $n.\frac{n-1}{2}a^{n-2} + n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}(h+k)$ + &c., therefore $(u) = n.\frac{n-1}{2}a^{n-2}$.

19. *In some cases the limits between which the value of* (u) *must lie can be determined.*

Ex. $u = \frac{c.(x-a)(y-b)}{(x-a)^2 + (y-b)^2}$, when $x = a$, $y = b$.

Substitute $x = a + h$, $y = b + k$, therefore $u = \frac{chk}{h^2 + k^2}$;

but $\frac{hk}{h^2+k^2}$ is not greater than $\frac{1}{2}$ and greater than $-\frac{1}{2}$, hence (u) lies between $\frac{c}{2}$ and $-\frac{c}{2}$.

20. *Conversely, if the ultimate value of a function of two variables be known, the ultimate ratio of the variables may be in some cases determined.*

Ex. Let the ultimate value of $\frac{2ax}{\sqrt{x^2+y^2}} = a$, when $x = 0$, $y = 0$, and let $n =$ the required ultimate ratio of $y : x$; then, since $u = \frac{2a}{\sqrt{1+\frac{y^2}{x^2}}}$ $\therefore (u) = \frac{2a}{\sqrt{1+n^2}} = a$ $\therefore n = \sqrt{3}$.

But implicit functions of two variables always appear in calculation under the form of $u = \text{F}(x, y) = 0$; and the object of the calculation is to determine the value of $\frac{dy}{dx}$ when it takes the form of $\frac{0}{0}$ in consequence of assigning particular values to the variables.

21. *Given* $u = \text{F}(x, y) = 0$; *required the value of* $\left(\frac{dy}{dx}\right)$ *which takes the form of* $\frac{0}{0}$.

$u = 0$ may be differentiated on the supposition that x and y are independent variables (4. 28.) or $du = \frac{du}{dx}dx + \frac{du}{dy}dy$;

whence $\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}}$, or $p = -\frac{u'}{u_{\prime}}$; which fails to give the value of p, for (ex hyp.) $u' = 0$ and $u_{\prime} = 0$; but since u' and $u_{\prime}$ are implicit functions of x and y, and that y is an explicit function of x, the value of $\frac{(u')}{(u_{\prime})}$ may be found by successive differentiations, as in Art. 6.; and by 4. 28. we shall have the same result, though not under the same form,

whether we differentiate $u'=0$ and $u_{\prime}=0$, on the supposition that they are functions of dependent or of independent variables; consequently, making the latter supposition, we have $(p) = \dfrac{(\text{P})dx + (\text{Q})dy}{(\text{R})dx + (\text{S})dy}$, where P, Q, R, S, are implicit functions of x and y, $= \dfrac{(\text{P}) + (\text{Q})\,(p)}{(\text{R}) + (\text{S})\,(p)}$, which gives an equation of the form $(p)^2 + \text{M}(p) + \text{N} = 0$, from which (p) may be determined.

If this value of (p) also $= \dfrac{0}{0}$, we must differentiate the numerator and denominator of the fraction $\dfrac{(\text{P}) + (\text{Q})\,(p)}{(\text{R}) + (\text{S})\,(p)}$ again, on the supposition that they are functions of two independent variables x and y, and dividing by dx, there will result a cubick for determining the value of $(p,)$ and so on.

If the original and consequently the derived functions, contain radicals, some of the quantities $(u)'$, $(u_{\prime})$, (u''), $(u'_{\prime})$, ... may become infinite, which would cause this method to fail; and to obviate this difficulty, we must begin the process with clearing $u = 0$ of radicals.

Cor. The number of values of (p) is equal to the number of differentiations required to obtain them. (Alg. 266.)

Examples.

Ex. 1. $ay^2 + x^3 - bx^2 = 0 = u$; to find the value of p when $x = 0$, and consequently $y = 0$.

$$p = \frac{2bx - 3x^2}{2ay} \therefore (p) = \frac{0}{0} = \frac{b}{a(p)} \therefore (p) = \pm\sqrt{\frac{b}{a}}.$$

Ex. 2. $y - b = (x-a)^{\frac{2}{3}}$; required p when $x = a$ and $y = b$. Clearing the equation of radicals, $(y-b)^3 - (x-a)^2 = 0$

$$= u \therefore p = \frac{2(x-a)}{3(y-b)^2} \therefore (p) = \frac{2}{6(y-b)} = \infty.$$

Ex. 3. $x^3 + y^3 - 3axy = 0$; required p when $x = 0$, $y = 0$.

$$p = \frac{ay - x^2}{y^2 - ax} = \frac{ap}{2yp - a} \therefore p(yp - a) = 0 \therefore \text{the two}$$

values of (p) are $(p) = 0$ and $(p) = \dfrac{a}{y} = \infty$.

Ex. 4. $x^4+ay^3=bx^2y$; required p when $x=0$, $y=0$.

$$p=\frac{2bxy-4x^3}{3ay^2-bx^2}\ \therefore (p)=\frac{bx(p)+by-b6^2}{3ay(p)-bx}=\frac{2b(p)}{3a(p)^2-b};$$

whence $(p)^3-\frac{b}{a}(p)=0\ \therefore (p)=0$, and $(p)=\pm\sqrt{\frac{b}{a}}$.

Ex. 5. $y^4-2axy^2+x^4=0$; required p when $x=0, y=0$.

$$p=\frac{ay^2-2x^3}{2y^3-2axy}\ \therefore (p)=\frac{ay(p)-3x^2}{3y^2(p)-ax(p)-ay}=\dots\dots$$

$\frac{a(p)^2}{6y(p)^2-2a(p)}$; whence $2y(p)^3-a(p)^2=0\ \therefore p=0, p=0$, and $p=\infty$.

Ex. 6. $y^4+2axy^2-ax^3=0$; required p when $x=0$, $y=0$.

The equation for determining p is $4yp^3+2ap^2-a=0$; hence one value of $p=\infty$, and the remaining two are $\pm\sqrt{\frac{1}{2}}$.

PRAXIS.

1. $y^2=x^2.\frac{x+a}{a}\ \therefore$ when $x=0$, $y=0$, $p=(1)^{\frac{1}{2}}$.

2. $y=\sqrt{-ax+x\sqrt{a^2+2ax-x^2}}\ \therefore$ when $x=0$, $y=0$, $p=\infty$, and $p=\pm1$.

3. $a^2y^3-bx^3y+x^5=0\ \therefore$ when $x=0, y=0, p^3=0$.

4. $y^4-2axy^2+x^4=0\ \therefore$ when $x=0$, $y=0$, $p=\infty$, and $p^2=0$.

5. $(y^2-x^2)^2+2ay^2x-x^3=0\ \therefore$ when $x=0$, $y=0$, $p^2=\frac{1}{2a}$ and $p=\infty$.

6. $(y^2+x^2)^2-6axy^2-2ax^3+a^2x^2=0\ \therefore$ when $x=0$, $y=0$, $p^2=\infty$.

CHAPTER VI.

Maxima and Minima functions of one or more variables.

1. *Def.* A variable magnitude is at its *maximum* value at that point where it ceases to increase and begins to decrease. It is a *minimum* where it ceases to decrease and begins to increase.

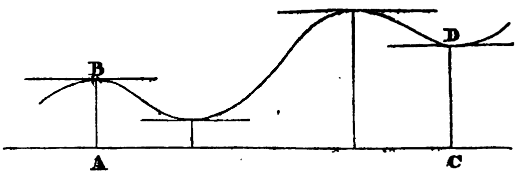
As the radius of a circle revolves, the sine increases through the first quadrant, and decreases through the second, and is therefore a maximum at the end of the first quadrant.

Cor. The reciprocal of a maximum is a minimum; and vice versâ.

2. An increasing negative quantity is considered to be a decreasing magnitude; thus the sine of $\frac{3\pi}{2}$, which is a negative maximum, must be considered as a *minimum*.

The radius vector of a parabola is a minimum at the vertex.

The same function may admit of several maxima or minima values, and it may happen that the maxima may be less than the minima, as appears from the annexed figure.



3. A decreasing positive quantity, before it becomes negative, passes through zero, and an increasing positive quantity passes through infinity; but these values have not the characteristick properties of a maximum or a minimum.

Thus the sine at the end of the second quadrant is not a minimum though it equals nothing; nor is the tangent at the end of the first quadrant, where it is infinite, a maximum in the technical sense of the word.

4. *Required to investigate the conditions of a maximum or a minimum.*

The essential character of a maximum consists in its being greater than either of the values which immediately precede and follow it. A minimum is less than either of its contiguous values.

Now, from Taylor's theorem (4. 5. Cor. 1.) we have

$$\text{U} = u + \frac{ph}{1} + \frac{qh^2}{1.2} + \frac{rh^3}{1.2.3} + \frac{sh^4}{1.2.3.4} + \&c.$$

$$u = u$$

$$_{\prime}u = u - \frac{ph}{1} + \frac{qh^2}{1.2} - \frac{rh^3}{1.2.3} + \frac{sh^4}{1.2.3.4} - \&c.$$

where h may be of any magnitude. But (3. 2.) h may be assumed so small that any term of this developement shall exceed the sum of the succeeding terms, and the same is true à fortiori for any less value of h; hence the three values of u, so far as regards their signs, may be expressed either

$$\left.\begin{aligned} \text{U} &= u + p\frac{h}{1} \\ u &= u \\ _{\prime}u &= u - p\frac{h}{1} \end{aligned}\right\} \text{or} \left\{\begin{aligned} \text{U} &= u + p\frac{h}{1} + q\frac{h^2}{1.2} \\ u &= u \\ _{\prime}u &= u - p\frac{h}{1} + q\frac{h^2}{1.2} \end{aligned}\right\} \text{or \&c.}$$

From the first, it appears that unless p is $= 0$, U and $_{\prime}u$ will be, the one greater, and the other less than u, or the function does not admit of a maximum or a minimum.

Next suppose, p being $= 0$, that q is not $= 0$, then we have

$$\left.\begin{aligned} \text{U} &= u + q\frac{h^2}{1.2} \\ u &= u \\ _{\prime}u &= u + q\frac{h^2}{1.2} \end{aligned}\right\}$$

where the contiguous values are necessarily both greater or both less than u, greater if q is positive, less if q is negative; and consequently, on this supposition there is a maximum or a minimum according as q is negative or positive.

Again, suppose $p = 0$, $q = 0$; but r not $= 0$; in this case there can be neither a maximum nor a minimum; but if $p = 0$, $q = 0$, $r = 0$, but s is not $= 0$, there will be a

maximum or a minimum according as s is negative or positive.

Generally, *there cannot be a maximum or a minimum unless the first fluxion vanishes, and unless the first of the fluxional coefficients which does not vanish is of an* even *order.*

5. *Required to find whether a proposed value of the principal variable causes the function to be a maximum or minimum.*

Let a be the proposed value of x. In $u = fx$, substitute $a+h$ and $a-h$ for x; then, h being indefinitely diminished, if $U - u$ and $,u - u$ have different signs, the function is not at a maximum or a minimum; but if they have the same sign, the function must be either a maximum or a minimum according as the sign is negative or positive.

Cor. If $U - u$ or $,u - u$ should contain $\sqrt{-1}$, the function cannot be at a maximum or a minimum; for an imaginary expression cannot be said to be either positive or negative. Alg. 354.

6. When a particular value is assigned to the variable the fluxional coefficients may take the form of $\frac{0}{0}$, or they may become infinite, (Ch. 5.) These cases will be more fully considered in the 12th chapter, as they cannot be understood without a knowledge of the theory of curves. But it is obvious that they are not included in the investigation Art. 4.; for the proposition upon which it is founded, viz. (3. 2.), does not obtain if any of the fluxional coefficients become infinite, or assume an indeterminate form. In these cases, the method pointed out in the preceding article will frequently enable us to ascertain whether the proposed value of the variable indicates a maximum or minimum state of the function.

If $(p) = \infty$ we may invert the function, and ascertain whether the proposed value of the variable indicates a maximum or minimum state of the inverted function. (Art. 1. Cor.)

7. *If* $u = fx$ *be at its maximum or minimum,* Au *where* A *is a constant quantity,* u^n, Lu *and* a^u *are at their maximum or minimum.*

For the first fluxional coefficients of these functions contain the factor p; and therefore $= 0$ when $p = 0$. In the

same manner it follows that their second fluxional coefficients $= 0$ when $q = 0$, and so on: consequently all these functions are at their maximum or minimum at the same time with u.

There is one exception with respect to the second and third of these functions.

Let v represent either of them; then, since $\text{v} = \text{F}u$ and $u = fx$, $\frac{d\text{v}}{dx} = \frac{d\text{v}}{du}\,\frac{du}{dx}$ (3. 8); if therefore $\left(\frac{d\text{v}}{du}\right) = \infty$, $\left(\frac{d\text{v}}{dx}\right)$ takes the form of $\frac{0}{0}$ (5. 13.) and may be finite.

Cor. The converse proposition does not hold good; that all the maxima and minima values of u^n belong also to u.

For $\frac{d\text{v}}{dx} = nu^{n-1}\frac{du}{dx} = 0$, which may be decomposed into $u^{n-1} = 0$, and $\frac{du}{dx} = 0$; of which the roots of the former are maxima or minima values of v, but not of u.

8. *If the equation $p = 0$ contains* n *equal roots, that root does or does not indicate a maximum or minimum, according as* n *is odd or even.*

For, let $p = (x-a)^n\text{P}$, where P does not contain $x - a$;

$$\text{then } q = n(x-a)^{n-1}\,\text{P} + (x-a)^n\frac{d\text{P}}{dx},$$

$$r = n(n-1)\,(x-a)^{n-2}\,\text{P} + 2n(x-a)^{n-1}\frac{d\text{P}}{dx} \;\ldots + (x-a)^n\frac{d^2\text{P}}{dx^2}.$$

&c. = &c.

Hence, if n be odd, the first fluxional coefficient which does not vanish, when x is substituted for a, is of an *even* order; and consequently (Art. 4.) a shows a maximum or a minimum. The reverse is the case when n is even.

The same proposition may be demonstrated of each of the equal roots, when $p = 0$ contains several different equal roots.

Cor. Since $q = 0$ is the limiting equation of $p = 0$ (Alg. 307.), the unequal roots of the latter, or the roots of $\text{P} = 0$, are alternately minima and maxima values in $u = 0$ (Alg. 298.).

9. The labour of finding the sign of (q) may sometimes be abridged by the following rule.

First, let p be under the form of a product.

Suppose $p = \mathrm{x}\,.\,\mathrm{x}'$ where x and x′ are functions of x, of which $(\mathrm{x}) = 0$; then, differentiating, we have $q = \frac{\mathrm{x}d\mathrm{x}'}{dx} + \frac{\mathrm{x}'d\mathrm{x}}{dx}$, therefore $(q) = \frac{1}{dx}(\mathrm{x}'d\mathrm{x})$; hence

Rule. *Multiply the fluxional coefficient of the factor which vanishes into the other factor; and in the result substitute for the variable its maximum or minimum value.*

Next, suppose $p = \frac{\mathrm{x}}{\mathrm{x}'}$, then $q = \frac{1}{dx}\left\{\frac{d\mathrm{x}}{\mathrm{x}'} - \frac{\mathrm{x}d\mathrm{x}'}{\mathrm{x}'^2}\right\}$, therefore $(q) = \frac{1}{dx}\left(\frac{\mathrm{x}'d\mathrm{x}}{\mathrm{x}'^2}\right)$; and since the sign of x′ in x'^2 cannot affect the result, the rule is the same as in the case of a product.

Ex. 1. $u = x^2.(x-a)^6$;

$\therefore p = 2x.(x-a)^6 + 6x^2.(x-a)^5 = 2x(x-a)^5(4x-a)$;

$\therefore p = 0$ when $x = 0$, $x = a$, and $x = \frac{a}{4}$.

For the first, $(q) = +2a^6$ or $x = 0$ indicates a *minimum*; when $x = \frac{a}{4}$, $(q) = 8x.(x-a)^5 = 2a\left(-\frac{3a}{4}\right)^5$, a *maximum*; when $x = a$, since the coefficient of $(x-a)^5$ is positive, $x = a$ indicates a *minimum*.

Ex. 2. Let $p = \frac{x-a}{x^{\frac{1}{2}}}$.

When $x = a$, $(q) = \frac{1}{a^{\frac{1}{2}}}$, which therefore indicates a *minimum*.

Ex. 3. $u = \frac{x^2-x+1}{x^2+x-1}$ or $p = \frac{2x^2-4x}{(x^2+x-1)^2} = \frac{2x(x-2)}{(x^2+x-1)^2}$.

When $x = 0$, $(q) = -4$; and when $x = 2$, (q) is positive, $\therefore$ we have $x = 0$, a *maximum*; and $x = 2$, a *minimum*.

10. *Implicit functions.*

Let the proposed function be $u = f(x, y) = 0$; then differentiating, $\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} = 0$, or $\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}}$, which $= 0$ when $\frac{du}{dx} = 0$.

Hence the two equations, $u = 0$ and $\frac{du}{dx} = 0$, will give the values of x and y, which correspond to a maximum or a minimum of y.

And to ascertain which it is, we must have recourse to the second fluxional equation $\frac{d^2u}{dx^2} + \frac{2d^2u}{dxdy}p + \frac{d^2u}{dy^2}p^2 + \frac{du}{dy}q = 0$ (4. 24.) which, since $p = 0$, becomes $\frac{d^2u}{dx^2} + \frac{du}{dy}(q) = 0$, from which the sign of q may be determined. If $q = 0$, there is not a maximum or minimum unless r also $= 0$, and we shall then have to calculate the sign of s from the fourth fluxional equation, and so on.

11. *Examples of maxima and minima.*

Ex. 1. Required the arc whose sine is a maximum.

$$\begin{array}{l|l} u = \sin.x & \\ p = \cos.x & \therefore \text{ when } x = \frac{\pi}{2}, p=0 \text{ and } q = -1, \text{ a } maximum, \\ q = -\sin.x & \end{array}$$

when $x = \frac{3\pi}{2}$, $p=0$, and $q = +1$, a *minimum.*

Ex. 2. Required x when $\sin.x + \cos.x =$ a maximum.

$$\begin{array}{l|l} u = \sin.x+\cos.x & \\ p = \cos.x-\sin.x & \therefore\ p=0 \text{ when } \cos.x = \sin.x \ \therefore\ x=\frac{\pi}{4}, \ \& \\ q = -\sin.x-\cos.x & \end{array}$$

$q = -\sqrt{2}$, which shows that $x = \frac{\pi}{4}$ is a maximum.

Ex. 3. $v.s\, x \times \cos.2x$ = maximum or minimum.

$u = (1-\cos.x)\cos.2x$

$= (1-\cos.x)(2\cos.^2x-1)$

$\therefore p = \sin.x(2\cos.^2x-1) - (1-\cos.x)\,4\cos.x\sin.x = 0 \;\therefore$ $6\cos.^2x - 4\cos.x - 1 = 0:\; \therefore \cos.x = \dfrac{\sqrt{2} \pm \sqrt{5}}{\sqrt{18}} =$, by substitution, b or $-\,c$.

Hence $\dfrac{p}{6} = \sin.x\,(\cos.x-b)\,(\cos.x+c) = 0$.

When $x=0$, $(\cos.x-b)\,(\cos.x+c) = +\dfrac{1}{6} \;\therefore (q) = +1$, or $x=0$ is a minimum.

When $\cos.x = b$, $(q) = -\sin.^2x\,(\cos.x+c)$, a maximum.

When $\cos.x = -c$, $(q) = -\sin.^2x(\cos.x-b) = \sin.^2x(b+c)$, a minimum.

In these examples the variable may have an infinite number of values.

Ex. 4. Required to divide a given $<$ A into x and y, so that $\sin.^mx \times \sin.^ny$ may be a maximum.

$x+y = \text{A} \;\therefore dx+dy=0$, or $dy = -dx$.

Also, $m\sin.^{m-1}x\cos.xdx \times \sin.^ny + n\sin.^{n-1}y\cos.ydy \times \sin.^mx = 0$, and substituting $-dx$ for dy, and dividing by $\sin.^{m-1}x\,\sin.^{n-1}ydx$, $m\cos.x\,\sin.y = n\cos.y\,\sin.x \;\therefore$ $m\dfrac{\sin.y}{\cos.y} = n\dfrac{\sin.x}{\cos.x}$, or $m\tan.y = n\tan.x \;\therefore m+n : m-n ::$ $\tan.x + \tan.y : \tan.x - \tan.y :: \sin.(x+y) : \sin.(x-y)$ $\therefore \sin.(x-y) = \dfrac{m-n}{m+n}\sin.\text{A}$, which determines x and y.

Ex. 5. Given $(x+y) = \text{A}$, and $m\cos.^2x + n\cos.^2y =$ a maximum; required x and y.

$2m\cos.x\sin.xdx + 2n\cos.y\sin.ydy = 0$ and $dx = -dy \;\therefore$ $m\cos.x\sin.x = n\cos.y\sin.y$, or $m\sin.2x = n.\sin.2y$ (Trig. p. 27.) $\therefore \sin.2x : \sin.2y :: n : m \;\therefore \tan.\text{A} : \tan.(x-y) :: n+m :$ $n-m$ or $\tan.(x-y) = \dfrac{n-m}{n+m}\tan.\text{A}$.

Ex. 6. To divide a given number into two parts, such that the product of the square of one part multiplied into the cube of the other may be a maximum.

Let $u = x^2.(a-x)^3 =$ maximum

$\therefore p = 2x.(a-x)^3 - 3x^2.(a-x)^2$

$= (2ax - 5x^2)(a-x)^2 = 0$

$= x.(2a-5x)(a-x)^2 = 0$, of which the factor $(a-x)^2$ does not belong to a maximum or minimum, because it contains an *even* number of equal roots.

Also $x = 0$ is a *minimum;* and $x = \frac{2a}{5}$, a maximum; or $\frac{2a}{5}$ and $\frac{3a}{5}$ are the two parts.

Ex. 7. Required the maxima and minima values of $u = \frac{\sqrt{a^2 + x^2}}{\sqrt{a + x}}$.

Let $v = u^2 = \frac{a^2 + x^2}{a + x}$, then (Art. 7.) the required maxima and minima must be included in those of v.

Now $\frac{dv}{dx} = \frac{2x}{a+x} - \frac{a^2+x^2}{(a+x)^2} = \frac{2ax + x^2 - a^2}{(a+x)^2}$; hence

$$p = \frac{\{x + (1 + \sqrt{2})a\}\{x + (1 - \sqrt{2})a\}}{(a+x)^2} = 0; \text{ of which}$$

$x = -(1 + \sqrt{2})a$ indicates a *maximum* of u^2; and $x = (\sqrt{2} - 1)\,a$ a *minimum* of u.

Ex. 8. Required the maxima and minima values of $u = x^m\,(a-x)^n$.

$p = mx^{m-1}(a-x)^n - nx^m(a-x)^{n-1} = x^{m-1}(a-x)^{n-1} \ldots \{ma - (m+n)x\} = 0.$

Hence (8) if $m - 1$ is even or m is odd, $x = 0$ gives neither a maximum nor a minimum: if m is even, since (q) is positive, $x = 0$ gives a minimum.

If n is odd, $x = a$ gives neither a maximum nor a minimum; if even, it gives a minimum.

Also, $x = \frac{ma}{m+n}$, which is intermediate between $x = 0$ and $x = a$, gives a maximum.

Ex. 9. Required the maxima and minima values of $\frac{x\sqrt{1-x^2}}{\sqrt{1-a^2x^2}}$.

Let $u = \frac{x^2(1-x^2)}{1-a^2x^2}$; if this were differentiated and made $=0$, the equation would rise above the second degree. Substitute $y=1-x^2$, $\therefore x^2=1-y$ & $1-a^2x^2=1-a^2+a^2y$ $=b+a^2y$ by substitution, $\therefore u=\frac{y-y^2}{b+a^2y}$ $\therefore \frac{du}{dy}=\frac{1-2y}{b+a^2y}$ $-\frac{(y-y^2)a^2}{(b+a^2y)^2}=\frac{b-2by-a^2y^2}{(b+a^2y)^2}=0$ $\therefore y=\frac{1-a^2\pm\sqrt{1-a^2}}{a^2}$

$\therefore x=\sqrt{\frac{-1\pm\sqrt{1-a^2}}{a^2}}$, an impossible quantity; which shows that the proposed function does not admit of a maximum or minimum.

Ex. 10. Required the maxima and minima values of $u=x^3-18x^2+96x-20$, x being supposed to increase.

Here $p=3x^2-36x+96=3(x-4)(x-8)=0$ $\therefore x=4$, and $x=8$.

$q=6x-36=6(x-6)$ $\therefore x=4$, a maximum; and $x=8$, a minimum.

If x is supposed to decrease, dx is negative, and the same result will be obtained.

Ex. 11. Let $u=x^3-2x^2y+3y^4=0$; required the maxima and minima values of y.

$$\left.\begin{aligned}&\frac{du}{dx}=3x^2-4xy=0=x(3x-4y)\\&\frac{du}{dy}=-2x^2+12y^3=0=2(6y^3-x^2)\\&\frac{d^2u}{dx^2}=6x-4y\end{aligned}\right\}\begin{aligned}&\therefore \text{(Art. 10.)}\ 6x-4y\\&+2(6y^3-x^2)(q)=0\end{aligned}$$

$\therefore (q)=\frac{2y-3x}{6y^3-x^2}$, or $y=\frac{3x}{4}$ is a maximum.

Also $y=0$ is not a maximum or minimum (Art. 4.).

Ex. 12. $a^4x^2=(x^2+y^2)^3$; required the maxima and minima values of y.

$la^4+2lx=3l(x^2+y^2)$ $\therefore \frac{dx}{x}=\frac{3(xdx+ydy)}{x^2+y^2}$, ...

$\therefore p=\frac{y^2-2x^2}{3xy}=0$ $\therefore y^2=2x^2$; which substitute in the

original equation, and there results $x^2 = \frac{\pm a^2}{\sqrt{27}}$ and $y = \pm a\sqrt{\frac{\pm 2}{\sqrt{27}}}$, two of which are maxima and minima values for y^2, but not for y; and of the remaining two the positive value is a minimum and the negative a maximum.

Ex. 13. Let $x^3 - axy + y^3 = 0$ be the equation of a curve; required the value of its greatest ordinate y.

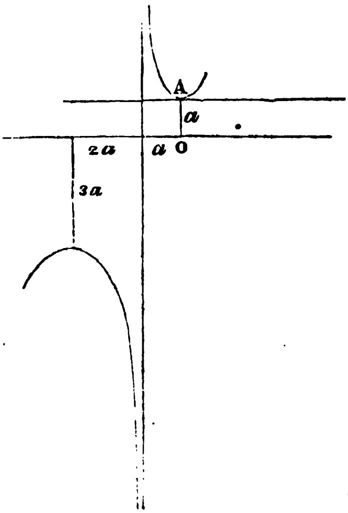
Differentiating $(3x^2 - ay)\,dx + (3y^2 - ax)\,dy = 0$ $\therefore$ $p = -\frac{3x^2 - ay}{3y^2 - ax} = 0$ $\therefore$ $3x^2 - ay = 0$ and $y = \frac{3x^2}{a}$; which substituted in the original equation gives $x = \frac{a\sqrt[3]{2}}{3}$ and $y = \frac{a\sqrt[3]{4}}{3}$; and applying the usual test, it will be seen that this is a *maximum* value of y.

Ex. 14. $y = \frac{a^3 - x^3}{a^2 - x^2}$; required the greatest and least ordinate.

$$p = \frac{-3x^2}{a^2 - x^2} + \frac{(a^3 - x^3)2x}{(a^2 - x^2)^2} = \frac{x^4 - 3a^2x^2 + 2a^3x}{(a^2 - x^2)^2}$$

$$= \frac{x(x-a)^2(x+2a)}{(a^2 - x^2)^2} = \frac{x(x+2a)}{(x+a)^2} = 0 \therefore x = 0 \;\&\; x = -2a.$$

$$q = \frac{2a^2}{(x+a)^3}.$$

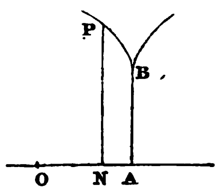


When $x = 0$, (q) is positive; hence, taking O for the origin, and drawing OA $= \frac{a^3}{a^2} = a$, A is a minimum point. When $x = -2a$, (q) is negative, which indicates a maximum; but y is now a negative magnitude, and therefore there is a negative minimum point whose co-ordinates are $-2a$ and $-3a$.

When $x=a$, p takes the form of $\frac{0}{0}$, but it does not indicate a maximum or minimum; which will appear from calculating the contiguous values of the function. (Art. 5.)

Ex. 15. $y-a=(a^3-2a^2x+ax^2)^{\frac{1}{3}}$, which is the equation of the cubical parabola.

$$p=\tfrac{2}{3}(a^3-2a^2x+ax^2)^{-\frac{1}{3}}(ax-a^2)$$
$$=\frac{2a(x-a)}{3a^{\frac{1}{3}}(a-x)^{\frac{4}{3}}}=\frac{0}{0}\ \ldots\ldots\ldots$$



when $x=a=-\dfrac{2a^{\frac{2}{3}}}{3(a-x)^{\frac{1}{3}}}=\infty$.

But, calculating the contiguous values, the differences will be found to be both positive; hence (Art. 5.) there is a *minimum* at $x=a$, $y=a$.

Ex. 16. $u=x\sqrt{ax-x^2}$

$$\therefore\ \mathrm{v}=u^2=ax^3-x^4\ \therefore\ \frac{d\mathrm{v}}{dx}=x^2(3a-4x)=0\ \therefore\ x=0,$$

$x=0$ and $x=\frac{3a}{4}$, of which, since $(q)=-$ ve, $x=\frac{3a}{4}$ not being a root of $u=0$ makes u a maximum. But $x=0$ is a root of $u=0$, and consequently it is possible that it may not indicate a maximum or minimum; and to determine this, find the contiguous values (Art. 4.), and it appears that the *preceding* value is impossible, or $x=0$ does not indicate a maximum or minimum of u.

Ex. 17. $u=\dfrac{a^2x}{(a-x)^2}$.

$$p=\frac{a^2}{(a-x)^2}+\frac{2a^2x}{(a-x)^3}=\frac{a^2(a+x)}{(a-x)^3}=0\ \therefore\ x=-a, \text{ and}$$

since (q) is positive, $x=-a$ indicates a minimum, which, since (u) is negative, may be considered as a negative maximum.

In this example $x=a$ gives $p=\infty$; hence, inverting the function, we have $\dfrac{d\frac{1}{u}}{dx}=\dfrac{-2(a-x)}{a^2x}-\dfrac{(a-x)^2}{a^2x^2}\ \ldots\ldots$

$=\dfrac{(x+a)(x-a)}{a^2x^2}$ $\therefore (q) = +$ ve, therefore $x=a$ makes $\dfrac{1}{u}$ a minimum, or $(u) = \infty$ = a maximum.

Ex. 18. $y = b + (x-a)^{\frac{5}{3}}$; to determine whether $x = a$ indicates a maximum or minimum.

$p = \dfrac{5}{3}(x-a)^{\frac{2}{3}}$ $q = \dfrac{10}{9}(x-a)^{-\frac{1}{3}}$	$\therefore$ when $x=a$, $p=0$, $q=\infty$; and calculating the contiguous values, we have $Y - y = +h^{\frac{5}{3}}$, and ${}_{\prime}y - y = -h^{\frac{5}{3}}$ or $x=a$ does not indicate a maximum or minimum.

Ex. 19. $y = b + (x-a)^{\frac{4}{3}}$.

$p = \dfrac{4}{3}(x-a)^{\frac{1}{3}}$ $q = \dfrac{4}{9}(x-a)^{-\frac{2}{3}}$	$\therefore$ when $x=a$, $p=0$, $q=\infty$; and proceeding as before $Y - y = +h^{\frac{4}{3}}$ and ${}_{\prime}y - y = +h^{\frac{4}{3}}$, or $x=a$ indicates a minimum, so that y cannot be less than b.

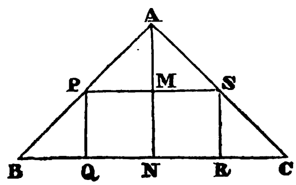
If $y = b - (x-a)^{\frac{4}{3}}$, y is at its maximum when $x = a$, and $y = b$.

Ex. 20. $u = x^x$; required u when a minimum.

$p = lx . x^x + x^x = x^x\{lx+1\}$ which $=0$ only when $lx = -1$, or $x = \dfrac{1}{e}$. $\therefore (q) = x^x \times \dfrac{1}{x}$, which is positive when $x = \dfrac{1}{e}$; hence $x = \dfrac{1}{e}$ gives a minimum value of u.

Ex. 21. To inscribe the greatest rectangle in a given triangle.

Let PQRS be the required rectangle in the triangle ABC. Draw AMN perpendicular to BC.

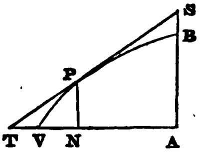


BC $=a$ AN $=b$ AM $=x$ $\therefore$ MN $=b-x$	$\therefore$ ($\triangle'$) AN : AM :: BC : PS $= \dfrac{ax}{b}$;

$\therefore u = \text{PS} \times \text{MN} = \dfrac{ax}{b}(b-x)$ or $= x(b-x)$ (Art. 7.)

$\therefore p = b - 2x = 0$, or $x = \dfrac{b}{2}$, and $(q) = -2$ which shows that the required rectangle bisects AN in M.

Ex. 22. Required the least triangle which can circumscribe a given parabolick area.



Let TAS be the required triangle touching the parabola in P, draw PN perpendicular to AV.

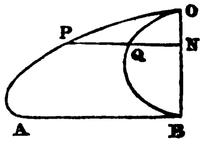
$\text{VA} = b$ $\text{VN} = x$ parameter $= a$	then (Con. Sect. Prop. 6 & 7) $\text{PN} = \sqrt{ax}$, & $\text{TN} = 2x$; $\therefore (\triangle^s)\ 2x : \sqrt{ax} :: b + x : \text{SA}$; which therefore $\propto \dfrac{b+x}{\sqrt{x}} \therefore \text{TAS} \propto \dfrac{(b+x)^2}{\sqrt{x}}$,

which is to be a maximum; hence $l.\dfrac{(b+x)^2}{x^{\frac{1}{2}}}$ or $2l(b+x) - \frac{1}{2}$

$lx = \text{maximum} = u$; $\therefore p = \dfrac{2}{b+x} - \dfrac{1}{2x} = \dfrac{3x - b}{2x(b+x)} = 0$;

$\therefore x = \dfrac{b}{3}$.

Ex. 23. Required the greatest ordinate of the *contracted* cycloid.



Its characteristick property is, that if OQB be a semicircle described upon its axis; and PQN any ordinate parallel to the base AB, OQ : PQ :: OQB : AB.

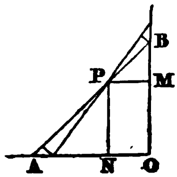
If $x = \text{ON}$, x shall $= \frac{1}{2}$ axis $+ \dfrac{1}{\pi}$ base.

Ex. 24. Required the point within a triangle, the sum of whose distances from the angles is a minimum.

Let P be the point within the triangle ABC; then, if AP be constant, its locus is a circle; suppose an ellipse described, foci B and C, touching this circle in P; it may be easily shown that BP + PC = a minimum; and since the curves have a common tangent, $\angle$ BPA = $\angle$ CPA. For the same

reason $\angle$ BPA $=$ $\angle$ BPC; or the required distances are equally inclined to each other.

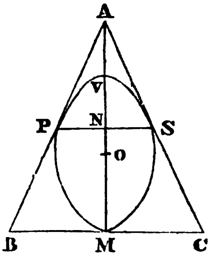
Ex. 25. Required the length of the shortest line which can be drawn through a point given in position between two rectangular axes.



Let a and b be the co-ordinates of P, then it may be shown that the required line $= \sqrt[3]{ab^2}$.

Results which appear under the form of maxima are sometimes minima. If it were required to find the longest ladder which can be moved under a given obstacle P along a vertical wall OB, the result would be the same as in this example.

Ex. 26. To inscribe the greatest ellipse in a given isosceles triangle.



Let O be the centre of the ellipse, x and y the semiaxes major and minor, AM perpendicular on BC the base of the triangle.

AM $= b$
MC $= a$ (Con. Sect. P. 13. Cor. 2.)

$$ON : OV :: OV : OA \quad \therefore ON = \frac{x^2}{b-x}.$$

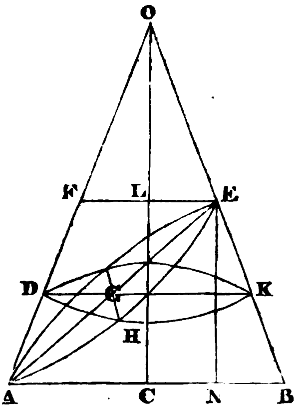
Hence $MN = \dfrac{bx}{b-x}$, $VN = \dfrac{(b-2x)x}{b-x}$ and $AN = \dfrac{(b-2x)b}{b-x}$.

Also, $NS = \dfrac{MC \times AN}{AM} = \dfrac{(b-2x)a}{b-x} \therefore \dfrac{(b-2x)^2.a^2}{(b-x)^2} = \dfrac{y^2}{x^2} \times \dfrac{(b-2x)bx^2}{(b-x)^2} \therefore y^2 = \dfrac{a^2}{b}(b-2x).$

But yx, and consequently $y^2x^2 =$ a maximum; hence $u = bx^2 - 2x^3 =$ maximum, $\therefore p = 2bx - 6x^2 = 2x(b-3x)$

$\therefore x = \dfrac{b}{3}$ and $y = \dfrac{a}{\sqrt{3}}$, and AN $= \frac{1}{2}$ AM.

Ex. 27. Required the greatest ellipse which can be cut from a given cone.



It is evident that the required section meets the base of the cone; let OAB be that section of the cone which is at right angles to the ellipse; AE the axis major; through its centre G draw a circle DHK parallel to the base of the cone, intersecting the ellipse in GH, which therefore is the semiaxis minor; draw OC, EN perpendicular and EF parallel to AB.

The area of the ellipse $\propto$ AE $\times$ GH (Con. Sect. 12. 7) $\propto$ AE $\sqrt{\text{DG} \times \text{GK}}$; but since AG = GE, DG = $\frac{1}{2}$ FE; and GK = $\frac{1}{2}$ AB a constant quantity; therefore the area $\propto$ AE $\times$ FE$^{\frac{1}{2}}$, or AE$^2 \times$ FE = maximum.

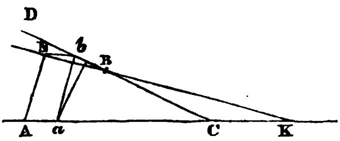
CN $= x$	$\therefore$ FE $= 2x$
OC $= b$	BN $= a-x$
AC $= a$	AN $= a+x$

OC : OL :: CB : LE and div°.
OC : LC = EN :: CB : BN, and EN $= \dfrac{\text{OC} \times \text{BN}}{\text{CB}}$; therefore AE^2 = AN^2 + NE^2 = $(a+x)^2 \ldots + \dfrac{b^2}{a^2}.(a-x)^2$; and $u = (a^2.(a+x)^2 + b^2.(a-x)^2)\,x =$ maximum, and $x = \dfrac{2a(b^2-a^2) \pm a(b^4 - 14a^2b^2 + a^4)^{\frac{1}{2}}}{3(a^2+b^2)}$.

If $a^4 - 14a^2b^2 + b^4$ be not positive, *i.e.* if the $\angle$ BOC be not less than 15°, the value of x is impossible; which shows that the greatest section possible is the base of the cone.

Draw AY perpendicular to OB, then as the section revolves from AO to AB the axis major decreases to AY, and afterwards increases, and the axis minor always increases; or the area must increase from AY: hence it follows, that if it admits of a maximum between AO and AY, it must also admit of a minimum. The same appears from calculating the value of (q).

See T. Simpson's Fluxions, who proves that unless $\angle$ BOC be less than 11°. 57′, the maximum ellipse will be less than the base of the cone.



Ex. 28. Two bodies start from A and B at the same time, and move with uniform velocities, which are $:: m : n$, and in directions AC, BD; required their position when their distance is a minimum.

T. Simpson (Flux. Ex. 13.) has given the following construction: Take CK : CB :: $m : n$; join KB; draw AN perpendicular to KB; draw Nb parallel to AC, meeting CBD in b; draw ba parallel to AN; and a and b are the required positions.

Ex. 29. Required the direction in which B must move, so that it may cross the line of A's direction at the greatest distance from it.

It appears from the sign of (q) that this result gives a maximum or a minimum according as B crosses before or behind A.

Ex. 30. Required the dimensions of a cylinder open at top, which, the solid content being given, shall have the least surface.

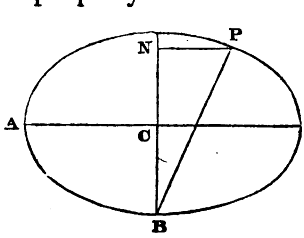
Let x = the radius of the base
y = the altitude

$\therefore y \times \pi x^2 = a$, and $y \times 2\pi x + \pi x^2 =$ minimum $\therefore 2yx + x^2 =$ minimum $\therefore \dfrac{2b}{x} + x^2 =$ minimum

$$= u \therefore p = -\frac{2b}{x^2} + 2x = 0 \therefore x = b^{\frac{1}{3}} \text{ and } y = \frac{b}{x^2} = b^{\frac{1}{3}}.$$

Or the radius of the base = the altitude.

Ex. 31. From the extremity of the axis minor of an ellipse draw the longest line to the periphery.



Let BP be the required line, draw PN an ordinate to the axis minor, and let BN $= x$, then $x^2 + \dfrac{a^2}{b^2}.(2bx - x^2) =$ maximum $= u$; $\therefore p = 2(x + \dfrac{a^2}{b} - \dfrac{a^2}{b^2}x) = 0 \therefore x = \dfrac{a^2 b}{a^2 - b^2}$.

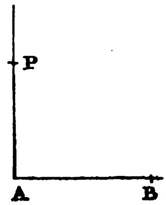
This example shows that the question is sometimes subject to restrictions from which the analytical result is wholly

free. In the present instance x cannot be greater than $2b$, and consequently the ellipse must be such that b is not greater than $\frac{a}{\sqrt{2}}$; a restriction which does not enter into the equation $x^2 + \frac{a^2}{b^2}(2bx - x^2) = \text{maximum}$.

When $b = \frac{a}{\sqrt{2}}$, and *à fortiori* when b approaches still nearer to a, the greatest line that can be drawn from B to the ellipse is the axis minor; but this result is not a *maximum*, because the principal variable cannot be made to increase so as to cause the contiguous values to be less than the intermediate value of the function.

T. Simpson's Fluxions.

Ex. 32.* PA is perpendicular to AB, which is given; required PA, so that the time of falling down PA + the time of describing AB with the last acquired velocity continued uniform may be a minimum.



$\left.\begin{array}{l}PA = x\\ AB = a\end{array}\right|$ (Mech. p. 60) the time down PA $= \sqrt{\frac{x}{m}}$; and with the velocity at A continued uniform twice PA would be described in this same time; hence $2x : a :: \sqrt{\frac{x}{m}}$: time through AB $= \frac{a}{2\sqrt{mx}}$; $\therefore \sqrt{\frac{x}{m}} + \frac{a}{2\sqrt{mx}}$, or $\sqrt{x} + \frac{a}{2\sqrt{x}} = \text{minimum} = u$; $\therefore p = \frac{1}{2x^{\frac{1}{2}}} - \frac{a}{4x^{\frac{3}{2}}} = 0$; $\therefore 2x = a$ and $x = \frac{a}{2}$.

Ex. 33. Given the base of an inclined plane; required its height, so that the time of descending along the plane + the time of describing the base with the last acquired velocity continued uniform may be a minimum.

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* If any of the following examples, taken from the elementary parts of Mechanics, are beyond the student's reading, they may be omitted for the present.

Let x = height a = base	then (Mech. p. 56. Cor. 5.) the time along

the plane $= \dfrac{\sqrt{a^2+x^2}}{\sqrt{mx}}$; $\therefore\ 2\sqrt{a^2+x^2} : a :: \dfrac{\sqrt{a^2+x^2}}{\sqrt{mx}}$: the time through the base $= \dfrac{a}{2\sqrt{mx}}$; $\therefore\ \dfrac{\sqrt{a^2+x^2}}{\sqrt{x}} + \dfrac{a}{2\sqrt{x}}$ = minimum = u.

$$\therefore\ u = \frac{2\sqrt{a^2+x^2}+a}{\sqrt{x}}.$$

$$\therefore\ p = \frac{2x^{\frac{1}{2}}}{\sqrt{a^2+x^2}} - \tfrac{1}{2}(2\sqrt{a^2+x^2}+a)x^{-\frac{3}{2}} = 0.$$

$$\therefore\ 2x^2 = a^2+x^2+\frac{a}{2}\sqrt{a^2+x^2}\ \therefore\ x^2-a^2 = \frac{a}{2}\sqrt{a^2+x^2}$$

$$\therefore\ x^4 - \frac{9a^2}{4}x^2 + \frac{3a^4}{4} = 0\ \therefore\ x = \sqrt{\frac{9+\sqrt{33}}{8}}\,a.$$

Ex. 34. Given the height of an inclined plane, required its length, so that a given weight acting upon another in a direction parallel to the plane, may draw it along the plane in the least time.

a = the height x = the length	Let P and W be the two bodies of which W is on the plane; then the moving force of

W $= \dfrac{a}{x}$W, and therefore the whole moving force $= \text{P} - \dfrac{a\text{W}}{x}$ $= \dfrac{\text{P}x - a\text{W}}{x}$; hence the accelerating force (Mech. Art. 24.) $= \dfrac{\text{P}x - a\text{W}}{x.(\text{P}+\text{W})}$, which therefore $\propto \dfrac{\text{P}x-a\text{W}}{x}$.

But $\text{S} \propto \text{FT}^2$ (Mech. p. 58.) $\therefore\ \text{T}^2 \propto \dfrac{\text{S}}{\text{F}} \propto \dfrac{x^2}{\text{P}x - a\text{W}}$ which = minimum $\therefore\ \dfrac{\text{P}x - a\text{W}}{x^2}$ or $\dfrac{\text{P}}{x} - \dfrac{a\text{W}}{x^2}$ = maximum = u

$$\therefore\ p = -\frac{\text{P}}{x^2} + \frac{2a\text{W}}{x^3} = 0\ \therefore\ x = \frac{2a\text{W}}{\text{P}}.$$

Ex. 35. Given two elastic bodies A and C, to find an intermediate body, x, so that the motion communicated from A to C through x may be a maximum. (Vince's Fluxions.)

a = A's velocity	then (Mech. p. 646.)
W = velocity communicated to x	$\left.\begin{array}{l} \text{A}+x : 2\text{A} :: a : \text{W} \\ x+\text{C} : 2x :: \text{W} : z \end{array}\right\}$
z = velocity communicated to z	

$$\therefore\ \text{A}x+x^2+\text{AC}+\text{C}x : 4\text{A}x :: a : z;$$

or $\text{A}+x+\dfrac{\text{AC}}{x}+\text{C} : 4\text{A} :: a : z$ which is to be a maximum;

consequently $\text{A}+x+\dfrac{\text{AC}}{x}+\text{C}$, or $x+\dfrac{\text{AC}}{x}$ = minimum $= u$;

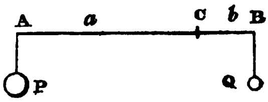
hence $p = 1-\dfrac{\text{AC}}{x^2} = 0$; or $\text{A} : x :: x : \text{C}$.

Ex. 36. P and Q are suspended over a fixed pulley, P is given and is greater than Q; required to find Q, so that P may communicate to it the greatest momentum in a given time.

Let $\text{Q} = x$, therefore (Mech. Art. 268.) $\dfrac{\text{P}-x}{\text{P}+x}$ = the accelerating force on Q, and therefore varies as the velocity generated in a given time; hence $\dfrac{\text{P}-x}{\text{P}+x}\times x \propto$ Q's momentum, or $u = \dfrac{\text{P}x-x^2}{\text{P}+x}$ = maximum $\therefore x = (\sqrt{2}-1)\text{P}$.

Ex. 37. P raises Q by means of a wheel and axle; find P so that it may communicate to Q the greatest momentum in a given time; the inertia of the machine being neglected.

AC $= a$	Let p be that part
BC $= b$	of P which Q sus-
Q $= x$	tains, or $a : b :: x :$



$p = \dfrac{b}{a}x$; $\therefore$ the moving force at $\text{P} = \text{P}-p = \text{P}-\dfrac{b}{a}x$.

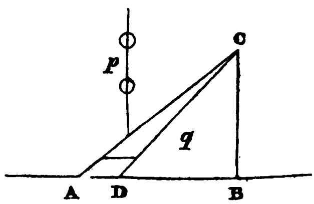
Also (Ch. 14.) $\dfrac{b^2}{a^2}x$ = the quantity of matter which must be substituted at A for Q, in order that the angular velocity may remain the same; hence the whole inertia at A = P

$+ \frac{b^2}{a^2} x$, and consequently the accelerating force at A (Mech. Art. 24.) $= \dfrac{\text{P} - \frac{b}{a} x}{\text{P} + \frac{b^2}{a^2} x}$, and its effect to generate angular velocity round C $\propto$ CA; hence the absolute accelerating force of the machine is as $a. \dfrac{\left(\text{P} - \frac{b}{a} x\right)}{\text{P} + \frac{b^2}{a^2} x} = \dfrac{a^2\text{P} - abx}{a^2\text{P} + b^2x}$ $\propto \dfrac{a\text{P} - bx}{a^2\text{P} + b^2x} \propto$ velocity generated in Q in a given time;

$\therefore u = \dfrac{a\text{P} - bx}{a^2\text{P} + b^2x} \times x =$ maximum $\therefore \dfrac{du}{dx} = \dfrac{a\text{P} - 2bx}{a^2\text{P} + b^2x} - \dfrac{(a\text{P}x - bx^2)b^2}{(a^2\text{P} + b^2x)^2} = \dfrac{a^2\text{P}^2 - 2a^2b\text{P}x - b^3x^2}{(a^2 + \text{P} + b^2x)^2} = 0$, from which x may be determined.

If we suppose the axle to be a solid cylinder, and take its inertia into the account, let its weight $=$ W; then, since the distance of the centre of gyration from the axis $= \frac{b}{\sqrt{2}}$ (Ch. 14), the quantity of matter which must be substituted for the axle at A $=$ W $\times \frac{b^2}{2a^2}$, and the whole inertia at A is now $\text{P} + \frac{b^2}{a^2} x + \frac{b^2}{2a^2}\text{W}$, and consequently the accelerating force of the machine is as $\dfrac{a^2\text{P} - abx}{a^2\text{P} + b^2x + \frac{1}{2} b^2\text{W}}$, and the problem may be solved as before.

Ex. 38. A rod of given weight (p), which passes through two small rings fixed in the same vertical line, by its pressure puts a solid inclined plane in motion along an horizontal table; given the weight of



the plane (q), find its elevation, so that its acceleration may be a maximum.

The only moving force is the weight (p), for (q) placed on an horizontal table possesses none; *i. e.* (q) cannot sustain any part of (p).

Let AB $= x$, BC $= y$, AC $= 1$ | Refer the motion to the rod p; let $v = p$'s velocity, then $y : x :: v : \frac{vx}{y} = q$'s velocity in its own direction; hence the inertia of q to retard $p = \frac{qx^2}{y^2}$;

and the accelerating force on $p = \frac{p}{p + \frac{qx^2}{y^2}}$, $\therefore$ the accelerating force on $q = \frac{px}{y\left(p + \frac{qx^2}{y^2}\right)} = \frac{pxy}{py^2 + qx^2}$, ...

$\therefore \frac{xy}{px^2 + qy^2} =$ maximum; or $u = \frac{py^2 + qx^2}{xy} = \frac{py}{x} + \frac{qx}{y}$ $=$ minimum, and $x^2 + y^2 = 1$, or $xdx + ydy = 0$; $\therefore$ $\frac{du}{dx} = -\frac{py}{x^2} + \frac{q}{y} - \frac{x}{y}\left(\frac{p}{x} - \frac{qx}{y^2}\right) = 0$; $\therefore$ transposing and multiplying by x^2y^3, $qx^2y^2 + qx^4 = px^2y^2 + py^4$, or $qx^2(x^2 + y^2)$ $= py^2(x^2 + y^2)$, $\therefore$ $qx^2 = py^2$ and $x : y :: \sqrt{p} : \sqrt{q}$, or tan. $\angle$ A $= \sqrt{\frac{q}{p}}$.

Ex. 39. A body (p) descends from rest down a solid inclined plane (q) moveable along an horizontal table; required the elevation of the plane, so that its acceleration may be a maximum.

Since the two bodies move freely, the whole *horizontal* momentum cannot be affected by their mutual action on each other (3d Law of Motion); hence, when p descends from rest, the whole horizontal momentum $= 0$, and consequently p's horizontal velocity : q's :: $q : p$, a constant ratio; or p descends in a right line.

AB $= a$, BC $= b$, AC $= l$ | Let p descend through CD while q moves through AD, then BD : AD :: p's horizontal velocity : q's ::

$q : p$ ∴ AB : AD :: $p + q : p$, or AD $= \frac{ap}{p+q}$. Similarly DB $= \frac{aq}{p+y}$, which determines the direction of CD.

And to find the accelerating forces on p and q; let $x =$ the pressure of p on $q = q$'s reaction;. resolving it into vertical and horizontal, we have $\frac{ax}{l} =$ the force to oppose p in the direction of gravity; hence $\frac{p - \frac{ax}{l}}{p} =$ the accelerating force on p in direction CB.

Again, $\frac{bx}{l} =$ the moving force either on p or q in an horizontal direction; therefore the accelerating force on $q = \frac{bx}{lq}$. But p's vertical velocity : q's velocity in the constant ratio of CB : AD, and consequently the accelerating forces in those directions are in the same ratio or . . . . . .

$\frac{p - \frac{ax}{l}}{p} : \frac{bx}{lq} :: b : \frac{ap}{p+q}$ ∴ $x = \frac{alpq}{l^2q + b^2p}$, and the accelerating force on $q = \frac{abp}{l^2q + b^2p}$, and the vertical accelerating force on $p = 1 - \frac{ax}{lp} = 1 - \frac{a^2q}{l^2q + b^2p} = \frac{b^2(p+q)}{l^2q+b^2p}$, and the whole accelerating force on p in the direction of the hypothenuse $= \frac{bl(p+q)}{l^2q+b^2p} = \frac{bls}{a^2q+b^2s}$ (if $s = p+q$.)

All these are uniform forces; hence, to find the elevation at which the acceleration is a maximum, we have, if $y =$ BC, $\frac{y}{a^2q + y^2s} =$ maximum, or $sy + \frac{a^2q}{y} =$ minimum $= u$ ∴ $\frac{du}{dy} = s - \frac{a^2q}{y^2} = 0$ ∴ $y = a\sqrt{\frac{q}{p+q}}$, & tan. $\angle$ A $= \sqrt{\frac{q}{p+q}}$.

This is John Bernoulli's problem, vol. 4. p.p. 332, who shows that if the body has any initial velocity at C, it will

describe a parabola. The same author has investigated the path described when the body descends down any given moveable curve.

Ex. 40. A machine, driven by the impulse of a stream, produces the greatest effect when the wheel moves with one-third of the velocity of the water. (Cambridge Problems, p. 356.)

Let a be the velocity of the water,

x the velocity of the wheel when the effect is a maximum.

Let A be the weight which the wheel can just sustain when its velocity is a.

Hence we have $a^2 : (a-x)^2 :: \text{A} : y$, the weight sustained when the velocity of the wheel is x; and to this weight the velocity x is communicated, consequently $u = xy = \frac{\text{A}}{a^2} x(a-x)^2 = \text{maximum}$; $\therefore \frac{du}{dx} = (a-x)^2 - 2x.(a-x) = 0 = (a-x)(a-3x)$; of which $x = a$ indicates a *minimum*, and $x = \frac{a}{3}$ a *maximum*.

Ex. 41. Required a circle such that the segment of an arc of given length may be a maximum.

$$\left.\begin{aligned}\text{Let } 2a &= \text{the arc}\\ 2x &= \text{the angle which it subtends}\end{aligned}\right\} \therefore \frac{a}{x} = \text{the radius.}$$

$$\text{Hence } u = \frac{a^2}{x} - \frac{a^2}{x^2}\sin.x\cos.x = \text{maximum},$$

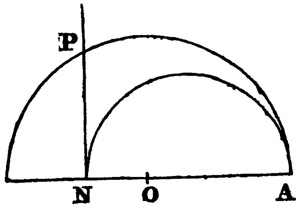
$$\therefore p = -\frac{a^2}{x^2} + \frac{a^2\sin.2x}{x^3} - \frac{a^2\cos.2x}{x^2} = \frac{a^2}{x^3}(\sin.2x \quad .\,. \\ - x\cos.2x - x) = 0.$$

When $x = \frac{\pi}{2}$, $p = 0$ and q is negative; therefore the area is a maximum when it is a semicircle.

When $x = 0$, or the radius is infinite, $p = \infty$; and this indicates a *minimum* (Art. 4.).

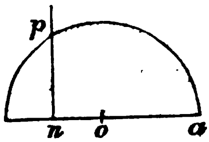
In this case the limit of the arc is a straight line coinciding with the chord.

Ex. 42. Required a circle such that the arc of a given versed sine may be a minimum.



Let AON be the given versed sine, and AO the required radius, and AP a minimum.

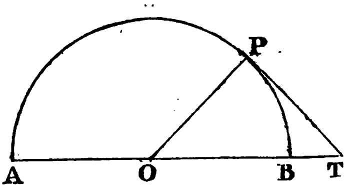
Draw a circular arc ap similar to AP, whose radius is $oa = 1$; let its versed sine be an.



$\text{AN} = a$ $ap = z$ $\text{AP} = u$	$\therefore\ u = \text{AP} = ap.\dfrac{\text{AN}}{an} = \dfrac{a.z}{vs.z} = \text{minimum}, \ \therefore \dfrac{du}{dz} = \dfrac{a}{vs.z} - \dfrac{az \sin.z}{vs^2 z} = 0;\ \therefore\ vs.z = z\sin.z,$ from

which equation the values of z, and therefore of AO, may be obtained by the method of trial and error.

Ex. 43. One end of a string of given length is fastened to a point A in the circumference of a circle, and wound round it so as to bring the other end T into the diameter AB produced, PT being a tangent at P; required the circle, so that the mixtilinear area PBT may be a maximum.



Describe the similar figure $apbt$, the radius $oa = 1$.

Let $\text{APT} = a$ $bp = z$ $\text{OB} = x$	then $2u = 2\text{PBT} = x^2(\tan.z - z) = \text{maximum};$ but $x\tan.z = \text{PT} = a - x(\pi - z)\ \therefore \tan.z - z = \dfrac{a - \pi x}{x}$; whence $2u = ax - \pi x^2;\ \therefore$

$0 = a - 2\pi x$, and $x = \dfrac{a}{2\pi}$.

12. The three last examples are remarkable in that the curve which forms the basis of the calculation is itself supposed to vary.

There is also a class of problems, called by the Bernoullis *Isoperimetrical*, in which the curve is not only indeterminate, but even its *species* is unknown. An instance of these is, "Required to determine the nature of a curve such that, its perimeter being given, it may include the greatest area."

In the fluxional calculus the form of the function, the variables of which are under consideration, is in general supposed to be fixed and determinate: if the function varies, and more especially if its form is unknown, the problem cannot be solved by the principles established in this chapter. In these cases we have to differentiate "de curvâ in curvam," to use Leibnitz' phrase; and they have given rise to a calculus which, by way of contradistinction, Euler and Lagrange have called the *Calculus of Variations*; though it forms part of the subject of fluxions, and may be established upon the same foundation, the principle of limits. We shall consider it in the second volume.

The first published Isoperimetrical problem, that of the solid of least resistance, was given by Sir I. Newton in the Principia, vol. ii. P. 34, Scholium.

13. "*Required to investigate the conditions of a maximum or minimum function of two independent variables.*"

Let $u = f(x, y)$, $\text{U} = f(x + h,\ y \pm k)$; then $_{,}u = f(x - h,\ y \mp k)$; and, as in Art. 4, the values of these functions, so far as regards their signs, may be expressed by (Vid. 4. 12.)

$$\left.\begin{aligned} \text{U} &= u + \frac{du}{dx}\frac{h}{1} \pm \frac{du}{dy}\frac{k}{1} \\ u &= u \\ {}_{,}u &= u - \frac{du}{dx}\frac{h}{1} \mp \frac{du}{dy}\frac{k}{1} \end{aligned}\right\}\text{or \&c.}$$

Now, first, in order that U and $_{,}u$ may be both greater or both less than u, a necessary condition is that $\frac{du}{dx}\frac{h}{1} \pm \frac{du}{dy}\frac{k}{1} = 0$; and h and k are independent quantities, consequently $\frac{du}{dx} = 0$, and $\frac{du}{dy} = 0$.

Next, take the third term in the developement of U, viz.

$\frac{1}{1.2}\left\{\frac{d^2u}{dx^2}h^2 \pm \frac{d^2u}{dxdy}2hk + \frac{d^2u}{dy^2}k^2\right\}$ which for the sake of brevity represent by $\frac{1}{1.2}\left\{\text{A}h^2 \pm \text{B}2hk + \text{C}k^2\right\}$.

Suppose that this function does not vanish; then (u) is or is not a maximum or minimum, according as the contiguous values of this function have the same or different signs; a maximum, if the sign is negative; a minimum, if positive.

We have then to investigate whether there can be any relation between A, B, C, such that the sign of the trinomial cannot be affected by any change in the signs of h and k.

Consider it as an equation, $\text{A}h^2 \pm 2\text{B}hk + \text{C}k^2 = 0$; then, if the roots of this equation are impossible, *i. e.* if AC be greater than B^2, whether h or k be considered as the unknown quantity, they lie under the form of the sum of two squares (Alg. 359.), and consequently the sign of the function cannot be affected by changing the sign either of h or of k; and (u) must be either a maximum or a minimum. And to determine which it is, since we may suppose $k = 0$, the trinomial is reduced to $\frac{1}{1.2}\text{A}h^2$; hence, AC being greater than B^2, (u) is a maximum or a minimum according as A is negative or positive.

It is obvious that A and C must have the same sign, otherwise AC cannot be greater than B^2.

Find then the corresponding values of x and y in the two equations $\frac{du}{dx} = 0$, $\frac{du}{dy} = 0$; substitute these values in A, C, and B; then if A and C have the same sign, either positive or negative, and AB be greater than C^2, there is a maximum or a minimum according as the sign of A and C is negative or positive; otherwise not.

If A, B, and C vanish, and also the fourth term of the developement; we shall have to investigate the conditions necessary to render the sign of a function of the form $\text{A}h^4 \pm 4\text{B}h^3k$ + &c. independent of the signs of h and of k; an investigation which hitherto has not been prosecuted with success.

If the variables are connected by an equation of condition; since $u = \text{F}(x, y)$ and $y = fx$, therefore $\frac{du}{dx} = \frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} = 0$; which reduces the problem to the case of a function of only one variable.

14. When u is a function of three or more independent variables $x, y, z \ldots.$ the conditions of a maximum or a minimum are that $\frac{du}{dx} = 0$, $\frac{du}{dy} = 0$, $\frac{du}{dz} = 0$, ... and if the third term of the expanded multinomial $\text{F}(x+h, y \pm k, z \pm l \ldots)$ does not vanish, we shall have to investigate the conditions in order that the sign of a function of the form $\text{A}h^2 + \text{B}k^2 + \text{C}l^2 \pm 2\text{A}hk \pm$ &c. may be independent of the signs of $h, k, l \ldots.$

These conditions are investigated in the *Fonctions Analytiques*, 2de Partie, Ch. 11.; but as they are complicated, the following simpler method should be adopted whenever it is practicable.

Since u is a function of independent variables, its maxima and minima may be found by supposing that all the variables except one are fixed and determinate. On this supposition we can find the relation which this variable bears to the rest; and pursuing the same method with all the variables, we shall have the equations necessary for determining their value. The principle of this method will be best illustrated by a geometrical example.

Ex. 1. Required to inscribe the greatest triangle in a given circle.

Let ABC be the required triangle; considering AB constant, ACB is greater than any other triangle as APB, from which it may be seen that AC = BC. For the same reason AC = AB; or the required triangle is equilateral.

Ex. 2. Required to inscribe in a circle a maximum polygon of a given number of sides.

Let ABCD be the required polygon, join AC; then the whole cannot be a maximum, unless a part as ABC is a maximum; whence it follows that the polygon is equilateral.

It may be observed, that when the proposed function is symmetrical with respect to all the variables, the value of all the variables may be deduced from the value of one of them.

$\frac{1}{1.2}\left\{\frac{d^2u}{dx^2}h^2 \pm \frac{d^2u}{dxdy}2hk + \frac{d^2u}{dy^2}k^2\right\}$ which ... riables which ...ion.

...at $u = \text{F}(x, y, z, t)$,

brevity represent by $\frac{1}{1.2}\left\{\text{A}h^2 \pm \text{B}2hk\right.$ ...tion, then we may ... x, y; and differen-

Suppose that this function does ... have two equations or is not a maximum or minimum ... guous values of this function h... tain $\frac{dz}{dx}, \frac{dt}{dx}$, and $\frac{dz}{dy}, \frac{dt}{dy}$ signs; a maximum, if the sign ... entiating the two equations positive. ... ult four equations containing

We have then to investi... relation between A, B, C, s... that there are altogether three cannot be affected by an...

Consider it as an eq... and three which contain $\frac{dz}{dy}, \frac{dt}{dy}$. if the roots of this ... greater than B^2, w... ion, there will arise two independent known quantity, ... ining these with the equations of con- squares (Alg. 3... are four equations to determine x, y, tion cannot be of k; and (u) ... example there are *three* equations of con- And to detern ... are only one principal variable, and *differen-* ... the three equations of condition on this sup-

the trinomial ... will result four equations between $\frac{dy}{dx}, \frac{dz}{dx}, \frac{dt}{dx}$,

than B^2, (u) ... negative or ... elimination, give one equation between *the vari-*

It is o... this, combined with the three equations of con- otherwise ... will enable us to determine the variables.

Find ... if any of the fluxional coefficients become infinite or ... minate, in consequence of assigning particular values equati... variable, we must adopt the same method as in the ... of a function of one variable.

and ... thus, let $x = a$, $y = b$, $z = c$, cause the developement or r... tail; then in the function u, substitute $x = a \pm h$, or $= b \pm k$, $z = c \pm l$, and finding the contiguous values or ... u and u when h, k, l, are indefinitely diminished, it will ... be seen whether these values of the variables give a maxi- mum or minimum value of the function.

Ex. $u = b - (x^2 + y^2)^{\frac{1}{3}}$ $\therefore \frac{du}{dx} = -\frac{\frac{2}{3}x}{(x^2+y^2)^{\frac{2}{3}}}$ and $\frac{du}{dy}$

$= -\frac{\frac{2}{3}y}{(x^2+y^2)^{\frac{2}{3}}}$, which when $x = 0$, $y = 0$ become $\frac{0}{0}$; but

be substituted for x and y, $u = b$ —

0, $y = 0$ indicate a *maximum*.

and minima functions of more

$z = a$ and $xy^2z^3 =$ maximum.

$y - z) - y^2z^3 = z^3y\,(2a - 3y - 2z) = 0$ $\therefore$

$$y = \frac{2.(a-z)}{3}.$$

$$\frac{}{dz} = 3y^2z^2(a-y-z) - y^2z^3 = z^2y^2(3a - 3y - 4z) = 0 \;\therefore$$

$$y = \frac{3a-4z}{3}, \text{ or } z = \tfrac{1}{2}a,\ y = \tfrac{1}{3}a \text{ and } x = \tfrac{1}{6}a.$$

$$\frac{d^2u}{dy^2} = z^3(2a - 3y - 2z) - 3z^3y = z^3(2a - 6y - 2z) \;\therefore$$

$$\text{A} = -a \times \frac{a^3}{8}.$$

$$\frac{d^2u}{dydz} = 3z^2y(2a - 3y - 2z) - 2z^3y = z^2y(6a - 9y - 8z) \;\therefore$$

$$\text{B} = -a \times \frac{a^3}{12}.$$

$$\frac{d^2u}{dz^2} = 2zy^2(3a - 3y - 4z) - 4z^2y^2 = 2zy^2(3a - 3y - 6z \;\therefore$$

$$\text{C} = -a \times \frac{a^3}{9}, \text{ where A} \times \text{C is greater than B}^2.$$

Ex. 2. Given $x + y + z = a$ and $x^m\,y^n\,z^r =$ maximum.

$$u = x^m y^n.\,(a - x - y)^r$$

$$\therefore \frac{du}{dx} = x^{m-1}y^n(a - x - y)^{r-1}\left\{ma - mx - my - rx\right\} = 0.$$

$$\frac{du}{dy} = x^m y^{n-1}(a - x - y)^{r-1}\left\{na - nx - ny - ry\right\} = 0.$$

It is obvious that the only factors to be considered are $ma - mx - my - rx = 0$, and $na - nx - ny - ry = 0$, which give $x = \dfrac{ma}{m+n+r}$, $y = \dfrac{na}{m+n+r}$, and $\therefore z = \dfrac{ra}{m+n+r}$,

Next, let u be a function of independent variables which are connected by certain equations of condition.

Taking a particular example; suppose that $u = \text{F}(x, y, z, t)$, and that there are *two* equations of condition, then we may consider z and t as implicit functions of x, y; and differentiating on this hypothesis, we shall have two equations $\frac{du}{dx} = 0$, and $\frac{du}{dy} = 0$, which contain $\frac{dz}{dx}, \frac{dt}{dx}$, and $\frac{dz}{dy}, \frac{dt}{dy}$ respectively. Also, by differentiating the two equations of condition, there will result four equations containing $\frac{dz}{dx}, \frac{dt}{dx}$, and $\frac{dz}{dy}, \frac{dt}{dy}$; so that there are altogether three which contain $\frac{dz}{dx}, \frac{dt}{dx}$, and three which contain $\frac{dz}{dy}, \frac{dt}{dy}$. Whence, by elimination, there will arise two independent equations, and combining these with the equations of condition, we shall have four equations to determine x, y, z, and t.

If in the same example there are *three* equations of condition we can have only one principal variable, and differentiating u and the three equations of condition on this supposition, there will result four equations between $\frac{dy}{dx}, \frac{dz}{dx}, \frac{dt}{dx}$, which, by elimination, give one equation between the variables; and this, combined with the three equations of condition, will enable us to determine the variables.

15. If any of the fluxional coefficients become infinite or indeterminate, in consequence of assigning particular values to the variable, we must adopt the same method as in the case of a function of one variable.

Thus, let $x = a$, $y = b$, $z = c$, cause the developement to fail; then in the function u, substitute $x = a \pm h$, $y = b \pm k$, $z = c \pm l$, and finding the contiguous values of U and ${}_{\prime}u$ when h, k, l, are indefinitely diminished, it will be seen whether these values of the variables give a maximum or minimum value of the function.

Ex. $u = b - (x^2 + y^2)^{\frac{1}{3}}$ $\therefore \frac{du}{dx} = -\frac{\frac{2}{3}x}{(x^2+y^2)^{\frac{2}{3}}}$ and $\frac{du}{dy} = -\frac{\frac{2}{3}y}{(x^2+y^2)^{\frac{2}{3}}}$, which when $x = 0$, $y = 0$ become $\frac{0}{0}$; but

if $0 \pm h$ and $0 \pm k$ be substituted for x and y, $u = b - (h^2 + k^2)^{\frac{1}{3}} = \mu$; or $x = 0$, $y = 0$ indicate a *maximum.*

16. *Examples of maxima and minima functions of more than one variable.*

Ex. 1. Given $x + y + z = a$ and $xy^2z^3 =$ maximum.

$$u = y^2z^3(a - y - z)$$

$$\therefore \frac{du}{dy} = 2z^3y(a-y-z) - y^2z^3 = z^3y\,(2a - 3y - 2z) = 0 \;\therefore$$

$$y = \frac{2.(a-z)}{3}.$$

$$\frac{du}{dz} = 3y^2z^2(a-y-z) - y^2z^3 = z^2y^2(3a - 3y - 4z) = 0 \;\therefore$$

$$y = \frac{3a - 4z}{3}, \text{ or } z = \tfrac{1}{2}a,\ y = \tfrac{1}{3}a \text{ and } x = \tfrac{1}{6}a.$$

$$\frac{d^2u}{dy^2} = z^3(2a - 3y - 2z) - 3z^3y = z^3(2a - 6y - 2z) \;\therefore$$

$$A = -a \times \frac{a^3}{8}.$$

$$\frac{d^2u}{dydz} = 3z^2y(2a - 3y - 2z) - 2z^3y = z^2y(6a - 9y - 8z) \;\therefore$$

$$B = -a \times \frac{a^3}{12}.$$

$$\frac{d^2u}{dz^2} = 2zy^2(3a - 3y - 4z) - 4z^2y^2 = 2zy^2(3a - 3y - 6z \;\therefore$$

$$C = -a \times \frac{a^3}{9}, \text{ where } A \times C \text{ is greater than } B^2.$$

Ex. 2. Given $x + y + z = a$ and $x^m\,y^n\,z^r =$ maximum.

$$u = x^my^n.\,(a - x - y)^r$$

$$\therefore \frac{du}{dx} = x^{m-1}y^n(a - x - y)^{r-1}\left\{ma - mx - my - rx\right\} = 0.$$

$$\frac{du}{dy} = x^m y^{n-1}(a - x - y)^{r-1}\left\{na - nx - ny - ry\right\} = 0.$$

It is obvious that the only factors to be considered are $ma - mx - my - rx = 0$, and $na - nx - ny - ry = 0$, which give $x = \frac{ma}{m+n+r}$, $y = \frac{na}{m+n+r}$, and $\therefore z = \frac{ra}{m+n+r}$,

which may be shown as in the last example to indicate a *maximum*.

Otherwise. Suppose z to be at its required value, and to be fixed; then, since $x+y+(z)=a$, and $x^m y^n z^r =$ maximum, therefore $x+y=b$ and $x^m y^n =$ maximum, and hence (11. Ex. 8.) $x:y::m:n$. Similarly it may be shown that $x:y:z::m:n:r$, which, combined with $x+y+z=a$, will give the required values of x, y, and z.

By this method it may be shown that when $x+y+z+t = a$, and $x^m y^n z^r t^s =$ maximum, $x = \dfrac{ma}{m+n+r+s}$,

$y = \dfrac{na}{m+n+r+s}$, &c. = &c.

Ex. 3. Required x, y, z, when $(b^3-x^3)(x^2z-z^3)(xy-y^2) =$ maximum $= u$.

Since there is no equation of condition we may suppose both z and x to be constant; hence $\dfrac{du}{dy} = x - 2y = 0$,

$\therefore y = \frac{1}{2}x$, $\therefore xy - y^2 = \frac{1}{4}x^2$, $\therefore (u) = \dfrac{x^2}{4}(b^3-x^3)(x^2z-z^3)$; hence, considering x constant, $x^2z - z^3 =$ maximum; $\therefore x^2 - 3z^2 = 0$, or $z = \dfrac{x}{\sqrt{3}}$, $\therefore x^2z - z^3 = \dfrac{2x^3}{3\sqrt{3}}$; or $(u) = \dfrac{x^2}{4} \cdot \dfrac{2x^3}{3\sqrt{3}}(b^3-x^3)$, $\therefore b^3x^5 - x^8 =$ maximum,

$\therefore x = \dfrac{b\sqrt[3]{5}}{2}$, $y = \dfrac{b\sqrt[3]{5}}{4}$, and $z = \dfrac{b\sqrt[3]{5}}{2\sqrt{3}}$, which, by applying the usual test, will indicate a *maximum*.

Ex. 4. $u = x^3 + y^3 - 3axy$; required the maxima and minima of u.

$$\frac{du}{dx} = 3(x^2 - ay) = 0$$

$$\frac{du}{dy} = 3(y^2 - ax) = 0$$

$$\frac{d^2u}{dx^2} = 6x \quad \therefore \text{A} = 0 \text{ or } 6a$$

$$\frac{d^2u}{dy^2} = 6y \therefore \text{c} = 0 \text{ or } 6a$$
$$\frac{d^2u}{dxdy} = -3a = \text{B}$$

$\therefore\ x = 0,\ y = 0$; or $x = a,\ y = a$.

Of these the first indicates neither a maximum nor minimum: the second indicates a minimum, or rather a negative maximum.

If a is negative, the function is $u = x^3 + y^3 + 3axy$, and its maximum value is $u = +a^3$.

Ex. 5. $u = (mx + n)(ny + m)$ = a maximum, and $a^{mx} b^{ny} = \text{c}$.

Here $u = \text{F}(x, y)$ and $y = fx$; and $\frac{dy}{dx} = -\frac{m}{n}\frac{la}{lb}$ $\therefore$

$\frac{du}{dx} = m(ny + m) - \frac{mla}{lb}.(mx + n) = 0$, $\therefore lb(ny + m) = la(mx + n)$, which, with the equation $mxla + nylb = lc$, will determine x and y.

Ex. 6. $x^4 z^2 y$ = maximum, and $x^2 + 2y^3 + z^4 = a$.

Here $u = 4lx + 2lz + ly$; and from the equation of condition $z = \text{F}(x, y)$;

$$\left.\begin{aligned}\therefore \frac{du}{dx} &= \frac{4}{x} + \frac{2}{z}\frac{dz}{dx} = 0 (1)\\ \frac{du}{dy} &= \frac{2}{z}\frac{dz}{dy} + \frac{1}{y} = 0 (2)\end{aligned}\right\}\text{also,}\quad \left.\begin{aligned}2x + 4z^3\frac{dz}{dx} &= 0 (3)\\ 6y^2 + 4z^3\frac{dz}{dy} &= 0 (4)\end{aligned}\right\}$$

Hence, eliminating $\frac{dz}{dx}$ from (1) and (3), and $\frac{dz}{dy}$ from (2) and (4), we have $\frac{4z}{2x} = \frac{2x}{4z^3}$ or $x^2 = 4z^4$; and $\frac{z}{2y} = \frac{6y^2}{4z^3}$ or $3y^3 = z^4$; whence, by substitution, $4z^4 + \frac{2}{3}z^4 + z^4 = a$;

$\therefore z^4 = \frac{3a}{17}$, $\therefore x^2 = \frac{12a}{17}$, and $y^3 = \frac{a}{17}$.

Ex. 7. Given $a^x b^y c^z = \text{A}$; required the maxima and minima of xyz.

$u = lx + ly + lz$; and $xla + ylb + zlc = l\text{A}$;

$$\left.\begin{aligned}\therefore \frac{du}{dx} &= \frac{1}{x} + \frac{1}{z}\frac{dz}{dx} = 0 \ (1) \\ \frac{du}{dy} &= \frac{1}{y} + \frac{1}{z}\frac{dz}{dy} = 0 \ (2)\end{aligned}\right\}$$

also, $$\left.\begin{aligned} la + \frac{dz}{dx} lc &= 0 (2) \\ lb + \frac{dz}{dy} lc &= 0 (3)\end{aligned}\right\} \therefore \begin{aligned}\frac{d^2z}{dx^2} &= 0 \\ \frac{d^2z}{dy^2} &= 0.\end{aligned}$$

Hence, by elimination, $x : y : z :: \frac{1}{la} : \frac{1}{lb} : \frac{1}{lc}$, which, combined with $xla + ylb + zlc = l\text{A}$, gives $x = \frac{l.\text{A}bc + la}{3la}$, $y = \frac{l.\text{A}ac + lb}{3lb}$ and $z = \frac{l.\text{A}ab + lc}{3lc}$.

And to determine whether these indicate a maximum or a minimum, we have

$$\left.\begin{aligned}\left(\frac{d^2u}{dx^2}\right) &= -\frac{1}{x^2} - \frac{1}{z^2}\frac{la^2}{lc^2} \\ \left(\frac{d^2u}{dy^2}\right) &= -\frac{1}{y^2} - \frac{1}{z^2}\frac{lb^2}{lc^2} \\ \left(\frac{d^2u}{dxdy}\right) &= -\frac{lalb}{z^2lc^2}\end{aligned}\right\}$$

from which it appears that $\frac{(l.\text{A}bc + la)(l.\text{A}ac + lb)(l.\text{A}ab + lc)}{27\, la\, lb\, lc}$ is a *maximum* value of xyz.

Ex. 8. Given $(x+1)(y+1)(z+1) = \text{A}$; required the maxima and minima of $a^x b^y c^z$.

(Cambridge Problems, p. 90.)

$u = la.x + lb.y + lc.z$; also, $\frac{dz}{dx} = -\frac{z+1}{x+1}$ & $\frac{dz}{dy} = -\frac{z+1}{y+1}$

$$\left.\begin{aligned}\therefore \frac{du}{dx} &= la - lc\frac{z+1}{x+1} = 0 \\ \frac{du}{dy} &= lb - lc\frac{z+1}{y+1} = 0\end{aligned}\right\}$$

$\therefore (x+1)\, la = (y+1)\, lb = (z+1) lc$, and $x+1 : y+1 : z+1 :: \frac{1}{la} : \frac{1}{lb} : \frac{1}{lc}$; whence, from the preceding example $x = \frac{l.\text{A}bc - 2la}{3la}$, $y = \frac{l.\text{A}ac - 2lb}{3lb}$ and $z = \frac{l.\text{A}ab - 2lc}{3lc}$.

$$\left.\begin{aligned}\text{Also, } \frac{d^2u}{dx^2} &= 2lc\frac{z+1}{(x+1)^2}\\ \frac{d^2u}{dy^2} &= 2lc\frac{z+1}{(y+1)^2}\\ \text{and } \frac{d^2u}{dydx} &= lc\frac{z+1}{(x+1)\,(y+1)}\end{aligned}\right\}\begin{array}{l}\text{which indicates a } \textit{minimum}\\ \text{value of the function.}\end{array}$$

Ex. 9. If a, b, c, be the prime factors of a number (N); required the number of the primes a, b, c, in order that N may admit of the greatest possible number of divisors.

Let x be the number of the a's, y of the b's, and z of the c's; then $\text{N} = \text{A}^x\,b^y\,c^z$; and since it is divisible by $1, a, a^2 \ldots a^x$, and by $1, b, b^2 \ldots. b^y$, and by $1, c, c^2 \ldots. c^z$; it is therefore divisible by every term of the continued product $(1+a+a^2+\ldots a^x)\,(1+b+b^2+\ldots b^y)\,(1+c+c^2+\ldots c^z)$; and the number of terms of this product $=(x+1)\,(y+1)\,(z+1)$; where we suppose that a, b, c, are all different numbers, and that N is a divisor of itself.

Hence $a^x\,b^y\,c^z = \text{N}$, and $(x+1)\,(y+1)\,(z+1) =$ maximum; or, multiplying the equation of condition by abc, we have $a^{x+1}\,b^{y+1}\,c^{z+1} = \text{N}$ and $(x+1)\,(y+1)\,(z+1) =$ maximum, and consequently, by Ex. 7, $x = \dfrac{l.\text{N}bc-2la}{3la}$, $y = \dfrac{l.\text{N}ac-2lb}{3lb}$, and $z = \dfrac{l.\text{N}ab-2lc}{3lc}$.

Ex. 10. Given the prime factors a, b, c, of a number N; required the relation between the indices, x, y, z, in order that the sum of the divisors may be a maximum.

It may be shown that the sum of the divisors $= \dfrac{a^{x+1}-1}{a-1}\cdot\dfrac{b^{y+1}-1}{b-1}\cdot\dfrac{c^{z+1}-1}{c-1}$; hence we have $u = l(a^{x+1}-1) + l(b^{y+1}-1) + l(c^{z+1}-1) =$ maximum, and $a^x\,b^y\,c^z = \text{N}$; whence $a^{x+1} = b^{y+1} = c^{z+1}$, or $x+1 : y+1 : z+1 :: \dfrac{1}{la} : \dfrac{1}{lb} : \dfrac{1}{lc}$, which combined with $xla + ylb + zlc = l\text{N}$, will give the relation between x, y, and z, and the required sum when a maximum may be found.

Ex. 11. Given four straight lines, required to form a trapezium such that the area may be a maximum.

Let a, b, c, e, be the sides of the trapezium,
$x =$ the angle made by a and e,
$y =$ that made by b and c,
then $u = ae \sin.x + bc \sin.y. =$ maximum.

Also, $a^2 + e^2 - 2ae \cos.x = b^2 + c^2 - 2bc \cos.y$, or $2ae \cos.x$

$2bc \cos.y = b^2 + c^2 - a^2 - e^2$, $\therefore \frac{dy}{dx} = \frac{ae \sin.x}{bc \sin.y}$.

Now $\frac{du}{dx} = \frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} = - ae \cos.x - ae \frac{\cos.y \sin.x}{\sin.y} = 0$,

$\therefore \tan.x = - \tan.y$, and $\therefore x + y = \pi$, or the trapezium may be inscribed in a circle; which is a property that may be easily deduced from geometrical principles. If then $s = a + b + c + e$, we shall have $u^2 = (s-a)(s-b)(s-c)(s-e)$.

Ex. 12. From a given point to draw the shortest line to a given sphere.

Take the given point for the origin; and placing the centre of the sphere in the plane zox (Vid. Ch. 7.), let its co-ordinates be a, o, b. Let x, y, z, be the co-ordinates of the required point of the sphere; then $x^2 + y^2 + z^2 = u^2 = \mathrm{v} =$ minimum; and the equation of condition is $(x-a)^2 + y^2 + (z-b)^2 = r^2$ where r is the sphere's radius. Hence

$$\left.\begin{array}{l} \frac{d\mathrm{v}}{dx} = x - z\frac{x-a}{z-b} = \frac{az - bx}{z-b} = 0 \\ \frac{d\mathrm{v}}{dy} = y - z\frac{y}{z-b} = \frac{-by}{z-b} = 0 \end{array}\right\} \text{whence } y = 0 \text{, or the least distance is situated in the plane zox.}$$

Also, $z = \frac{bx}{a}$, and by elimination $(x-a)^2\left(1 + \frac{b^2}{a^2}\right) = r^2$,

$\therefore x - a = \frac{\pm ar}{\sqrt{a^2 + b^2}}$, of which the $+$ indicates a *maximum* and the $-$ a *minimum*.

Ex. 13. To inscribe the greatest paralleliped in a sphere.

Let $2x$, $2y$, $2z$, be the sides of the solid, then $u = xyz =$ maximum; and $x^2 + y^2 + z^2 = r^2$ is the equation of condition; $\therefore \frac{du}{dx} = yz - yx$, which $= 0$ when $x = z$; and the variables are symmetrical with respect to each other,

$\therefore x = y = z$, and $u = \dfrac{8r^3}{3\sqrt{3}}$, which may be shown to be a *maximum* value of u.

If it be required to inscribe the greatest paralleleliped in a given spheroid, the equation of condition is $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 + \left(\dfrac{z}{c}\right)^2 = 1$; and the required solid $= \dfrac{8abc}{3\sqrt{3}}$.

Ex. 14. Two bodies move in opposite directions with velocities, the sum of which is given. Show that the sum of the products of each body into the square of its velocity is a *minimum*, when the velocities are reciprocally proportional to the quantities of matter in the bodies.

Let A and B be the bodies, x and y their velocities; then $u = Ax^2 + By^2$ and $x + y = c$. $\therefore \dfrac{dy}{dx} = -1$, $\therefore$ $\dfrac{du}{dx} = 2Ax - 2By = 0$, $\therefore x : y :: B : A$; and since $\dfrac{d^2u}{dx^2} = 2A + 2B$, these values indicate a *minimum*.

Ex. 15. Required the magnitudes of three perfectly elastic balls x, y, z, which, when interposed between two given balls a and b, will cause a moving with a given velocity to communicate to b the maximum velocity.

Let $v = a$'s velocity, then (Mech. Art. 204) $a + x : 2a :: v : \dfrac{2av}{a+x} = x$'s velocity. Similarly, if we calculate the velocity of y, z, and b, we shall find b's velocity $= \dfrac{16avxyz}{(a+x)(x+y)(y+z)(z+b)}$, and here we may suppose both y and z to be constant; $\therefore \dfrac{x}{(a+x)(x+y)} =$ maximum, or $\dfrac{x^2 + (a+y)x + ay}{x}$, or $x + \dfrac{ay}{x} =$ minimum; $\therefore$ $1 - \dfrac{ay}{x^2} = 0$, or $a : x :: x : y$. In the same manner it may be shown that $a : x :: x : y :: y : z :: z : b$. Also, since

$x+\frac{ay}{x}=u$, $\therefore \frac{du}{dx}=1-\frac{ay}{x^2}$, $\therefore \frac{d^2u}{dx^2}=\frac{2ay}{x^3}$, which shows that $x^2=ay$ is a *minimum*, and the same may be proved true of y and z.

Ex. 16. Required the least distance between two given right lines in space.

Let $\left.\begin{array}{l} x = az+\alpha \\ y = bz+\beta \end{array}\right\}$ and $\left.\begin{array}{l} x = cz+\kappa \\ y = ez+\varepsilon \end{array}\right\}$ be the equations in space of the two given lines (Ch. 7. Art. 7.).

Let x', y', z', and x'', y'', z'', be the co-ordinates of the extreme points of the required line; then $\mathrm{v} = u^2 = (x'-x'')^2 + (y'-y'')^2 + (z'-z'')^2 =$ minimum (7. 41.); and for these *six* variables, there are *four* equations of condition, viz. $\left.\begin{array}{l} x' = az'+\alpha \\ y' = bz'+\beta \end{array}\right\}$ and $\left.\begin{array}{l} x'' = cz''+\kappa \\ y'' = ez''+\varepsilon \end{array}\right\}$, and consequently we may consider v as a function containing *two* independent principals z' and z''; and we have

$$\frac{d\mathrm{v}}{dz'} = 2a(x'-x'') + 2b(y'-y'') + 2(z'-z'') = 0,$$

$$\frac{d\mathrm{v}}{dz''} = -2c(x'-x'') - 2e(y'-y'') - 2(z'-z'') = 0,$$

which, combined with the four equations of condition, will determine the value of u.

These equations belong to a *minimum*, for

$$\left.\begin{array}{l} \dfrac{d^2\mathrm{v}}{dz'^2} = 2(a^2+b^2+1) \\ \dfrac{d^2\mathrm{v}}{dz''^2} = 2(c^2+e^2+1) \\ \dfrac{d^2\mathrm{v}}{dz'dz''} = -2(ac+be+1) \end{array}\right\}$$

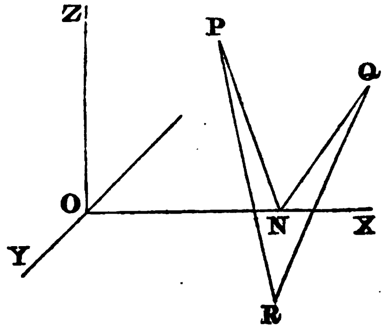
and $(a^2+b^2+1)(c^2+e^2+1) - (ac+be+1)^2 = (a-c)^2 + (b-e)^2 + (ae-bc)^2$, and is therefore > 0.

The required line is at right angles to each of the given lines.

For, let $\left.\begin{array}{l} x = mz+\mu \\ y = nz+\nu \end{array}\right\}$ be the equations of the required line; then, since it passes through the points x', y', z', and x'', y'', z'', we have $\left.\begin{array}{l} x'-x'' = m(z'-z'') \\ y'-y'' = n(z'-z'') \end{array}\right\}$ and consequently, by

substitution, and dividing by $\pm 2(z'-z'')$, $\left.\begin{array}{r} am+bn+1=0 \\ cm+en+1=0 \end{array}\right\}$
which are the equations of condition that the lines may be at right angles to each other (7. 42.).

Ex. 17. Two points and a plane are given in position; it is required to find the shortest path between the points and the plane.



Through the given points P and Q draw a plane perpendicular to the given plane; take these for the co-ordinate planes ZOX, YOX; and let the required path be PRQ.

Let x' o, z', & x'', o, z'', be the given co-ordinates of P & of Q,
x, y, o, the required co-ordinates of R;

then PR + RQ $= \sqrt{(x-x')^2+y^2+z'^2} + \sqrt{(x''-x)^2+y^2+z''^2}$ $=$ minimum $= u$, which is a function that contains only two variables x and y.

Hence $\dfrac{du}{dx} = \dfrac{x-x'}{\sqrt{(x-x')^2+y^2+z'^2}} - \dfrac{x''-x}{\sqrt{(x''-x)^2+y^2+z''^2}} = 0 \quad (1)$

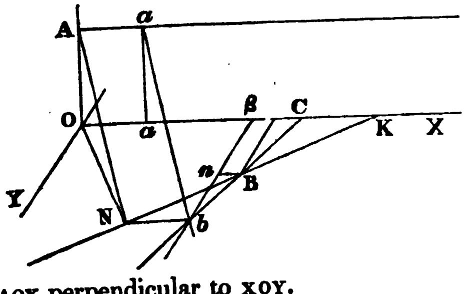
$$\frac{du}{dy} = \frac{y}{\sqrt{(x-x')^2+y^2+z'^2}} + \frac{y}{\sqrt{(x''-x)^2+y^2+z''^2}} = 0 \quad (2)$$

From (2) we have $y = 0$, or the shortest path lies in a plane perpendicular to the given plane.

From (1) we have $\dfrac{x-x'}{\sqrt{(x-x')^2+z'^2}} = \dfrac{x''-x}{\sqrt{(x''-x)^2+z''^2}}$

i. e. if PNQ is the path, cos. ∠ PNO = cos. ∠ QNX, or ∠ PNO = ∠ QNX.

Ex. 18. Required the least distance of two points which move from given positions and describe given lines in space with known uniform velocities.



Let A and B be their initial positions, Aa, Bb, the lines they describe, ab the required least distance.

Draw through Bb a plane XOY parallel to Aa; and through Aa draw AOX perpendicular to XOY.

Take O for the origin, and let o, o, a, and b, c, o, be the given co-ordinates of A and B. Produce bB to meet OX in C, and let $\angle$ OCB $= \alpha$.

Let A's velocity : B's : : $m : n$.

Also let x', o, a, be the co-ordinates of a ; x'', y'', o, those of b ; then $V = u^2 = (x''-x')^2 + y''^2 + a^2$.

Draw the co-ordinates of B and b, and Bn perpendicular to $b\beta$ the ordinate of b;

$$\left.\begin{array}{l}\text{then } \mathrm{A}a = x' : \mathrm{B}b :: m : n, \\ \quad\text{and } \mathrm{B}b : \mathrm{B}n :: 1 : \cos.\alpha\end{array}\right\} \therefore \mathrm{B}n = \frac{n\cos.\alpha}{m}x', \;\ldots\ldots$$

$$\therefore x'' = b - \frac{n\cos.\alpha}{m}x'.$$

$$\left.\begin{array}{l}\text{Also,} \quad x' : \mathrm{B}b :: m : n, \\ \quad\text{and } \mathrm{B}b : bn :: 1 : \sin.\alpha\end{array}\right\} \therefore bn = \frac{n\sin.\alpha}{m}x', \;\ldots\ldots$$

$$\therefore y'' = c + \frac{n\sin.\alpha}{m}x'.$$

Hence, by substitution,

$$\mathrm{V} = \left(b - \frac{n\cos.\alpha + m}{m}x'\right)^2 + \left(c + \frac{n\sin.\alpha}{m}x'\right)^2 + a^2;$$

$$\therefore \frac{d\mathrm{V}}{2dx'} = -\frac{n\cos.\alpha + m}{m}\left(b - \frac{n\cos.\alpha + m}{m}x'\right) + \frac{n\sin.\alpha}{m}\left(c + \frac{n\sin.\alpha}{m}x'\right); \text{ wherefore } x' = \frac{m^2b + mn(b\cos.\alpha - c\sin.\alpha)}{m^2 + 2mn\cos.\alpha + n^2}.$$

In the same manner the values of x'' and of y'' may be calculated, which determine the position of the required line.

The following construction has been given for determining the points a and b.

In CX take CK : CB :: $m : n$; join KB and draw ON perpendicular on KB produced; draw Nb parallel to OX, meeting CB in b, and joining AN, draw ba parallel to AN.

(Vid. Garnier's Calc. Diff. p. 411.)

Ex. 19. The equation of curved surfaces of the second degree is $z^2 = ax^2 + 2bxy + cy^2 - ex - fy + g$; required to investigate the conditions that z may be a maximum or minimum.

Let $u = z^2$ then

$$\left.\begin{aligned} \frac{du}{dx} &= 2ax + 2by - e = 0 \\ \frac{du}{dy} &= 2cy + 2bx - f = 0 \end{aligned}\right\} \therefore\ x = \frac{ec - fb}{2(ac - b^2)},\ y = \frac{eb - fa}{2(ac - b^2)}$$

$$\left.\begin{aligned} \frac{d^2u}{dx^2} &= 2a \\ \frac{d^2u}{dy^2} &= 2c \\ \frac{d^2u}{dxdy} &= 2b \end{aligned}\right\}$$ whence a condition is that a and c have the same sign, and that ac is greater than b^2.

PRAXIS.

1. Required the arc whose sin. $\times$ v.s. $=$ maximum. $x = 120°$.

2. Required the angle the excess of whose sine above its versed sine is a maximum. $x = 45°$.

3. $x + y = \text{A}$, and $\tan.^m x. \tan.^n y =$ maximum; $\therefore \tan. (x - y) = \dfrac{m - n}{m + n} \tan. \text{A}$.

4. $x + y = \text{A}$, and $\sec.^m x \sec.^n y =$ maximum; $\therefore \sin. (x - y) = \dfrac{n - m}{n + m} \sin. \text{A}$.

5. $x + y = \text{A}$, and $\sin.^m x \cos.^n y =$ maximum or minimum.

6. Required the fraction whose excess above its cube is a maximum; $x = \sqrt{\frac{1}{3}}$.

7. Divide a given number into two parts such that their product × their difference may be a maximum.

$x = \dfrac{\pm 1 + \sqrt{3}}{2\sqrt{3}} a$ indicates a maximum.

8. $u = \dfrac{x\sqrt{a^2 - x^2}}{\sqrt{a^2 + x^2}}$; $\therefore x = a$ or $x = 0$ is not a maximum or minimum: $x = a(-1 + \sqrt{2})^{\frac{1}{2}}$ is a maximum.

9. $u = \dfrac{a^2x^2 + 4b^4x}{(b^2 - ax)^3}$; $\therefore x = -\dfrac{2b^2}{a}$, a maximum; $x = -\dfrac{2b^2}{3a}$, a minimum.

10. $u = \dfrac{x^5 - ax^4}{(x^2 - 4a^2)^2}$; $\therefore x = 0$, a maximum; $x = 4a$, a minimum; $x = 2a(-1 + \sqrt{2})$, a maximum.

11. $u = \dfrac{x^{n+1}}{2a^n} + \dfrac{a^n}{2x^{n-1}}$; $\therefore x = \pm\left(\dfrac{n-1}{n+1}\right)^{\frac{1}{2n}} \cdot a$, of which, if n is odd, both indicate a minimum; if n is even, the positive a minimum and the negative a maximum.

12. $u = \cos.x + \cos.2x + \cos.3x$; $\therefore x = 0$, a maximum; $\cos.x = \dfrac{-1 \pm \sqrt{7}}{6} a$, a minimum and a maximum.

13. $2axy = a^3 + ax^2 - bx^2$; $\therefore x = \dfrac{\pm a^{\frac{3}{2}}}{\sqrt{a - b}}$, two minima of y.

14. $u = 3x^4 - 28ax^3 + 84a^2x^2 - 96a^3x + 48b^4$; $\therefore x = a$, a minimum; $x = 2a$, a maximum; and $x = 4a$, a minimum.

15. $2axy = a^3 + (a - b)x^2$. There is a positive and a negative minimum; and their co-ordinates are $x = \dfrac{\pm a^{\frac{3}{2}}}{\sqrt{a - b}}$, $y = \pm\sqrt{a^2 - ab}$.

16. $y = b \pm (x - a)^2$; required to ascertain whether $x = a$ indicates a maximum or minimum. When the sign is positive, a minimum; when negative, a maximum.

17. $u = x^{\frac{1}{x}}$; $\therefore x = e$, a maximum.

18. Into how many equal parts must a given quantity Q be divided in order that their continued product may be a maximum? $x = \frac{Q}{e}$.

19. $u = \frac{e^{\cos. x}}{\cos.^n x}$; $\therefore$ $x = \cos.^{-1} n$, a minimum; $x = 0$, a minimum.

20. Inscribe the greatest parallelogram in a parabola; and also in a double Cissoid of Diocles. $x = \frac{1}{3}a$; and

21. Inscribe the greatest parabola in the segment of a circle, the chord being a tangent to the vertex of the parabola.

22. In the space included between two concentrick circles inscribe the greatest parallelogram. $x = \frac{r + \sqrt{8R^2 + r^2}}{4}$.

23. Through a point P given in position within the $\angle$ O to draw the shortest line. Also draw APB so that AO + OB = a minimum.

$x = \sqrt{ab}$ and $x^3 - cx^2 + bcx - a^2b = 0$ where a and b are the co-ordinates of P.

24. In a line joining two luminous points of given intensity, find the point which is the least illumined. $x = \frac{a\sqrt[3]{\alpha}}{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$.

25. Two equal fires are placed in the foci of an ellipse; required the point in the periphery where the heat is the least. The extremity of the axis minor.

26. The inclination of the conjugate diameters of an ellipse is a maximum, when they are equal.

27. Inscribe the greatest parabola in a given triangle. $x = \frac{a}{4}$.

28. Required the greatest parabola which can be cut from a given cone. $x = \frac{3a}{4}$.

29. From a given cone, required to cut a parabola so that the solid generated by its revolution, either about its axis or its greatest ordinate, may be a maximum. $x = \frac{2}{3}b$: and $x = \frac{5}{6}b$.

30. The area of the least parabola which can circumscribe a given circle is one whose latus rectum is equal to the radius of the circle.

31. Find A which impinges upon B at rest, so that when reflected its momentum may be a maximum. $x = b(-1 + \sqrt{2})$.

32. Given the base of an inclined plane; required its height so that the time of falling down the height + the time of describing the base with the last acquired velocity continued uniform may be a minimum. $x = \frac{1}{2}a$.

33. Find a point in a circle whose plane is vertical, so that the time of falling to an horizontal tangent and of describing it with the last acquired velocity may be a minimum. $x = \dfrac{8r}{5}$.

34. P raises Q on an inclined plane by a string parallel to the planes; required Q so that P may communicate to it the greatest momentum in a given time.

35. P raises Q by means of a wheel and axle; given P and Q and the radius of the axle; required the radius of the wheel, so that the acceleration of Q may be a maximum; the inertia of the machine being neglected. . . . .

$$b = \frac{a}{p}\left\{q + \sqrt{q^2 + pq}\right\}.$$

36. P and Q are equal: P raises Q through a given space by means of a wheel and axle in the least time possible; required the ratio of the radii of the wheel and of the axle.

37. Materials are to be raised through a given altitude by a given wheel and axle; required the quantity to be raised at each ascent in order that a maximum may be raised in a given time; the inertia of the machine being neglected.

$$u = \frac{a\text{P}x^2 - bx^3}{a\,\text{P} + b^2x} = \text{maximum}.$$

38. Of all cones of revolution under a given surface, required that whose content is a maximum. $y^2 = 2x^2$.

39. Of all cones of revolution of the same content, required that which has the *whole* of its surface a minimum. $y^2 = 8x^2$.

40. $u = ax^2 - bxy + czx + yz$ has not a maximum or minimum at $x = 0,\ y = 0,\ z = 0$.

41. Of all parallelelipeds of the same content, required that which has the least surface. $x = y = z = a$.

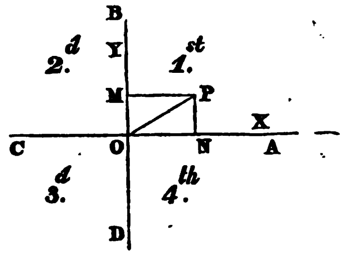
42. $u = x^2y^3z^4$ a maximum, and $2x + 3y + 4z = a$;
$\therefore\ x = \frac{a}{9},\ y = \frac{a}{9},\ z = \frac{a}{9}$.

43. $u = x^4yz^2$ a maximum, and $x^2 + 2y^3 + z^4 = a$;
$\therefore\ x^2 = \frac{12a}{17},\ y^3 = \frac{a}{17},\ z^4 = \frac{3a}{17}$.

Before we proceed to apply the principles of the Calculus to Curves it will be necessary to establish a few propositions relating to Analytick Geometry, which shall form the subject of the following chapter.

CHAPTER VII.

Lines and Surfaces.



1. The *position* of a point in a plane is determined by referring it to two known axes intersecting at any angle, usually a right angle; thus AOC, BOD, being two rectangular axes, and P being a point within the ∠ AOB, draw PN, PM, perpendicular to OA, OB; then the position of P depends upon the magnitudes of PN and PM, which are called its *rectangular co-ordinates.*

Since ON = PM, ON and NP are also the co-ordinates of P; and of these, when they are referred to the axis OA, ON is the abscissa and NP the ordinate; and vice versâ when they are referred to the axis OB.

COA is the axis of the abscissas or the axis X, BOD the axis of the ordinates or Y.

2. There are four points, one in each of the four right angles, whose co-ordinates are of the same magnitude; but the ambiguity is removed by attending to their signs. Thus, in the first, the signs are + +; in the second, − +; in the third, − −; and in the fourth + −: all of which are different.

3. *Def.* The *equation of a line* is that which expresses the relation between the co-ordinates of any point in it. It is usually deduced from some geometrical property which characterizes the line.

Irregular curves, or those curves which are not described by some certain law, are never considered in science, as they cannot become subjects of calculation.

If the equation is not algebraick, the curve is said to be *transcendental* or *mechanical.*

4. *Def.* A *diameter* is that axis x which to any the same abscissa has the sum of the positive ordinates equal to the sum of the negative. (Vid. Alg. 521.)

The *axis* of the curve is that diameter which is at right angles to its ordinates.

5. *Def.* There are certain lines in curves called *parameters*. They are the lines upon which the *magnitude* of the whole curve depends, and which are independent of its position. If they are changed, no change is produced in the nature or properties of the curve itself. Circles, parabolas, cycloids, and catenaries, have only one parameter. Ellipses and the trochoids have two.

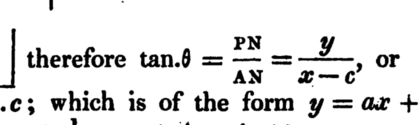
6. *Def.* 1. When the ordinates are similar and equal on both sides of the axis x, the curve is said to be *symmetrical* on the axis x. In this case the whole curve will balance itself upon the axis in all positions. (Mech. Art. 86.)

Cor. When the equation is such that, changing the signs of the co-ordinates, it remains unaltered, the curve is symmetrical in the two opposite quadrants.

Def. 2. When this obtains, the origin is called the *centre* of the curve.

7. *The equation of a right line is of the form $y=ax+\alpha$.*

For let BA be the line cutting the axis x in the point A, and making with it the angle BAN.



Take any point P in AB, and draw PN perpendicular to OAN.

$$\left.\begin{array}{ll} \text{ON} = x, & \angle\,\text{PAN} = \theta \\ \text{NP} = y & \text{OA} = c \end{array}\right| \text{ therefore } \tan.\theta = \frac{\text{PN}}{\text{AN}} = \frac{y}{x-c}, \text{ or}$$

$y = \tan.\theta . x - \tan.\theta . c$; which is of the form $y = ax + \alpha$, where $a = \tan.\angle\,\text{BAN}$, and $\alpha = \text{OA}.\tan.\angle\,\text{BAO}$.

If the axes are not rectangular, but inclined at an $\angle\,\omega$, the equation is of the same form, but in this case $a=\dfrac{\sin.\theta}{\sin.(\omega-\theta)}$, and $\alpha = -\dfrac{\sin.\theta . c}{\sin.(\omega-\theta)}$.

Cor. The equation is also of the form $\dfrac{x}{\pm a} + \dfrac{y}{\pm b} = 1$, where $a = \text{OA}$, $b = \text{OB}$. (Fig. 10.)

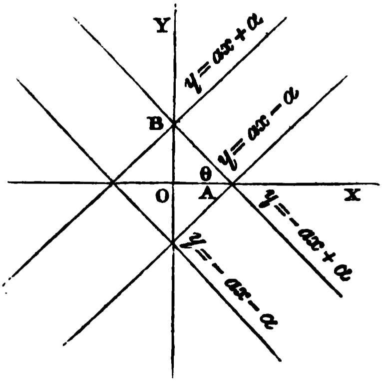
8. Conversely, a simple equation of the form $y = ax + \alpha$ belongs to a right line which intersects the axis X at an angle whose trigonometrical tangent $= a$, and at a distance from the origin $= -\dfrac{\alpha}{a}$.

For there cannot be two right lines drawn from the same point in the axis and making the same angle with it.

9. *Def.* The quantities a and α are called the *arbitrary constants* of the equation; because they may be assumed at pleasure, and by assigning proper values to them, the equation may be made to belong to any proposed right line.

Since the equation contains two arbitrary constants, a right line may be drawn fulfilling two conditions. It may be made to pass through two given points, or it may be drawn through a given point making a given angle with a given right line.

10. It appears from the annexed figure that we may have four right lines the equations of which shall contain the same arbitrary constants; but the ambiguity will also be removed in this case by attending to the signs of the constants. Those lines which intersect each other in the axis X have the signs of their arbitrary constants reversed.



11. *Required the equation of a straight line which is at right angles to a given straight line.*

In fig. 7 draw PG at right angles to AB, cutting the axis in G; then, since tan. ∠ PGX = − tan. ∠ PGA = − cot. ∠ PAG $= -\dfrac{1}{a}$, the required equation is of the form $y = -\dfrac{1}{a}x + \beta$, where β remains indeterminate, since it requires another condition to fix the *position* of PG.

Cor. Conversely, if we have two equations $y = ax + a$,

and $y = bx + \beta$; and $b = -\frac{1}{a}$, or $a = -\frac{1}{b}$, the lines to which these equations belong are at right angles to each other.

12. *Required the equation of a straight line drawn from a given point at right angles to a given straight line.*

Let $\left.\begin{array}{l} y = ax + \alpha \\ y = bx + \beta \end{array}\right\}$ be the equations of the given and of the required lines, and x', y' the co-ordinates of the given point; then it may be shown as in (11.) that $b = -\frac{1}{a}$; therefore the equations are $\left.\begin{array}{l} y = ax + \alpha \\ y = -\frac{1}{a}x + \beta \end{array}\right\}$, and since the line passes through the point x', y', we have $y' = -\frac{1}{a}x' + \beta$; whence, eliminating β, $y - y' = -\frac{1}{a}(x - x')$, which is the required equation.

13. *From the same data, required the equation of a line drawn from the given point which shall make the same angle with the perpendicular line which the perpendicular line makes with the ordinate.*

Let PK (fig. 7.) make the $\angle$ KPG = the $\angle$ NPG; its equation being of the form $y = a'x + \alpha'$, it may be shown as before, since it passes through the point x', y', that $y - y' = a'(x - x')$; but a' here $= -\tan.\angle$ PKN $= -\cot.2\angle$ NPG $= -\cot.2\angle$ PAN $= \frac{a^2 - 1}{2a}$, wherefore the required equation is $y - y' = \frac{a^2 - 1}{2a}(x - x')$.

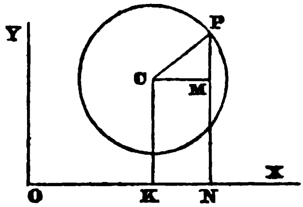
This is the equation of a linear caustick where APB is the reflecting line.

14. *Curves.*

Ex. 1. Required to find the equation of the circle.

Its characteristick property is that all its radii are equal.

Let C be its centre, draw its co-ordinates OK, KC; and also the co-ordinates ON, NP, of any point P in the circumference. Draw CM perpendicular to PN and join CP.



$$\left.\begin{array}{ll} OK = a, & ON = x \\ KC = b, & NP = y \end{array}\right\} CP = r.$$

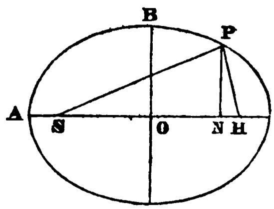
Then (Eu. 47. 1.) $CP^2 = CM^2 + MP^2$, or $r^2 = (x-a)^2 + (y-b)^2$ is the required equation.

Cor. If the origin of the co-ordinates coincides with the centre of the circle, the equation becomes $r^2 = x^2 + y^2$, or $y^2 = r^2 - x^2$, or $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$.

The equation of the circle contains *three* arbitrary constants, two of which determine its position and the third its magnitude. Hence a circle may be described fulfilling three conditions; it may be drawn through three given points, or touching three given straight lines.

Ex. 2. Required the equation of an *ellipse*, the origin being at the centre.

The characteristick property of this curve is, that the sum of two lines, SP, HP, drawn from fixed points S and H is invariable. Draw PN perpendicular to SH.



$$\begin{array}{r|l} SP + HP = 2a & \text{then } \left.\begin{array}{l}(c+x)^2 + y^2 = z^2 \\ \text{and } (c-x)^2 + y^2 = (2a-z)^2\end{array}\right\} \ldots\ldots\ldots \\ ON = x & \\ NP = y & \therefore 4cx = -4a^2 + 4az, \text{ or } z = \frac{cx}{a} + a; \therefore \\ SO = c & \\ SP = z & \\ a^2 - c^2 = b^2 & y^2 = \left(\frac{cx}{a} + a\right)^2 - (c+x)^2 = a^2 - c^2 - \end{array}$$

$(a^2 - c^2)\frac{x^2}{a^2} = b^2 - \frac{b^2}{a^2}x^2$, or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, or $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$.

If AO be the semi-axis major $= a$, and A be taken as the origin; for x substitute $x - a$, and the equation becomes $y^2 = \frac{b^2}{a^2}(2ax - x^2)$.

If $a = \infty$, $y^2 = \dfrac{b^2}{a^2}2ax = \dfrac{2b^2}{a}x = \text{L} \times x$, the equation of a parabola.

In the hyperbola HP − SP $= 2a$, and its equations are $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, or $y^2 = \dfrac{b^2}{a^2}(2ax + x^2)$, according as the origin is at the centre or at the vertex.

If $b = a$ the hyperbola is equilateral or rectangular; and its equations are $y^2 = x^2 - a^2$, and $y^2 = 2ax + x^2$.

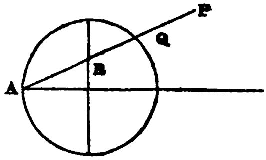
Def. If AO : SO :: 1 : e, e is called the eccentricity of the conick section; hence SO $= ae$; and in the ellipse $b^2 = a^2 - a^2e^2$, or the equations of the ellipse may be put under the form $y^2 = (1 - e^2)(a^2 - x^2)$, and $y^2 = (1 - e^2)(2ax - x^2)$.

In the hyperbola, SO^2 = AO^2 + BO^2, or $a^2e^2 = a^2 + b^2$, and its equations are $y^2 = (e^2 - 1)(x^2 - a^2)$, and $y^2 = (e^2 - 1)(2ax + x^2)$. (Con. Sect. p. 69, Art. 6.)

Cor. 1. When $e = 0$, the equations of the ellipse belong to a circle whose radius $= a$.

Cor. 2. When the hyperbola is equilateral, $e = \sqrt{2}$.

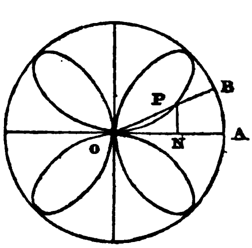
Ex. 3. AP is drawn from one extremity of a quadrant cutting the radius of the other extremity in R, and the circle in Q, and is produced so that QP = RQ; required the equation of the curve traced by P.



Place the origin at the centre, and the equation is

$$y = (a + x)\frac{\sqrt{2a - x}}{\sqrt{2a + x}}.$$

Ex. 4. As the radius OB $= a$ of a circle revolves, OP is always taken equal to $a.\sin.2\angle$ AOB; required the equation of the curve traced by P.



Let $\theta = \angle$ AOB, then $\dfrac{y}{x} = \tan.\theta$;

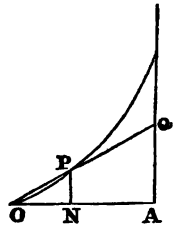
$$\therefore \frac{y^2}{x^2} = \sec.^2\theta - 1 = \frac{a^2\sin.^2 2\theta}{x^2} - 1$$

$=\dfrac{4a^2y^2}{(x^2+y^2)^2}-1$; $\therefore (x^2+y^2)^3=4a^2x^2y^2$, which is the required equation.

15. *Conversely when the equation of a curve is given, it can in some cases be constructed geometrically.*

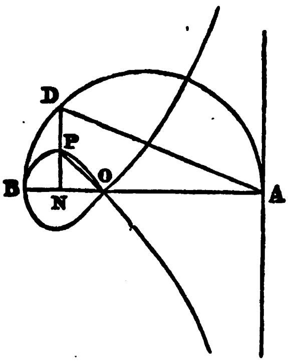
Ex. 1. Given $y^2=ax$; required to construct the curve.

Take $\left.\begin{matrix}\text{OA}=a\\ \text{ON}=y\end{matrix}\right\}$ Always take AQ perpendicular to OA equal to ON; join OQ, draw NP at right angles to OA, meeting OQ in P, and P will trace the required curve.



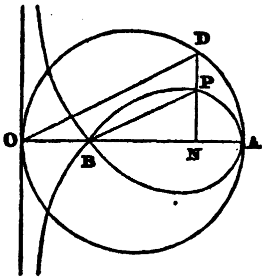
Ex. 2. Given $y^2=\dfrac{x^3+bx^2}{c-x}$, required to construct the curve.

Since $y^2=x^2\dfrac{x+b}{c-x}$, $y:x::\sqrt{x+b}:\sqrt{c-x}$; take therefore OA $=c$, OB $=b$; and upon AB describe a semicircle; take in OB, ON always $=x$; erect the ordinate ND; join AD and draw OP making the $\angle$ NOP $=\angle$ ADN, and cutting ND in P; P shall trace the required curve.

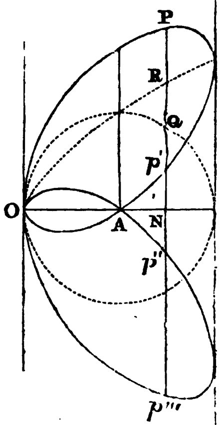


Ex. 3. $xy^2+x^3-(2b+a)x^2+(b^2+2ab)x-ab^2=0$.

In OA $=a$ take OB $=b$; upon OA describe a circle, draw any ordinate ND, join OD and draw BP parallel to OD, cutting ND in P, which is a point in the required curve.



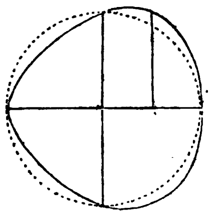
Ex. 4. $y^4+2x^2y^2+x^4-6axy^2-2ax^3+a^2x^2=0$; $\therefore y^4+(2x^2-6ax)y^2+(x^2-3ax)^2=-4ax^3+8a^2x^2=4ax(2ax-x^2)$; $\therefore y^2=3ax-x^2\pm\sqrt{4ax}\sqrt{2ax-x^2}=ax\pm\sqrt{4ax}\sqrt{2ax-x^2}+2ax-x^2$; $\therefore y=\pm\sqrt{ax}\pm\sqrt{2ax-x^2}$.



Hence describe a circle OQ and a parabola OR whose radius and parameter $=a$; draw any ordinate NQR and take NP and NP''' = NR + NQ, and NP', NP'' = NR − NQ, and the four points of the curve will be found.

PRAXIS.

1. $y^2=x\sqrt{2ax-x^2}$ may be constructed by means of a circle and a parabola.



2. $(x^2-a^2)^2+(y^2-a^2)^2=0$ is an equation to two ellipses whose centres coincide, and whose axes majores are at right angles to each other.

3. $y^4-2x^2y^2+x^4-a^2x^2=0$ may be decomposed into $(y^2-x^2+ax)(y^2-x^2-ax)=0$, and consequently belongs to two curves whose equations are $y^2-x^2+ax=0$, and $y^2-x^2-ax=0$, the circle and the rectangular hyperbola.

16. If the equation can be solved with respect to either of the variables, it is evident that the position of the points of the curve may be arithmetically computed.

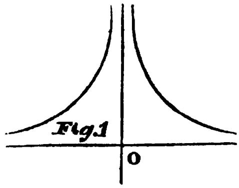
It is frequently useful in computing a curve from its equation to assume a third variable whose relation to one of the co-ordinates is simpler than the proposed relation.

Thus, let $y^4+x^2y^2+2y^3-x^3=0$; to obtain either x or y as explicit functions, we must solve an equation of the third or fourth degree. Assume then $x=yz$, therefore, by substitution, and dividing by y^3, $y+yz^2+2-z^3=0$, or $y=\frac{z^3-2}{z^2+1}$; and assuming $z=0$, $z=1$, $z=2$, &c. we can compute the corresponding values of x and y.

17. The equation will also enable us to ascertain the limits within which the curve must be included, and to give the position of its infinite branches; and since, by the assistance of the calculus, as will be shown in the following chapters, we can determine the convex and the concave portions of the curve, the magnitude and position of its greatest and least ordinates, and the situation of what are called the *singular points* of the curve, we are enabled, even without constructing or computing it, to ascertain its general outline and form. This method of *tracing* out the curve, as it is called, will be best understood from the examples.

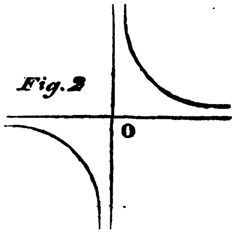
Ex. 1. Let the curve be of the hyberbolick species whose equation is $y = \dfrac{a^{n+1}}{x^n}$.

If n is even the curve is fig. 1; if odd, fig. 2.



Ex. 2. $y^2 = \dfrac{x^3 + bx^2}{c - x}$, or . . . . .

$y = \pm x \sqrt{\dfrac{b+x}{c-x}}$. Vid. fig. 15. Ex. 2.



For every positive value of y, there is an equal negative value, hence the curve is symmetrical on the axis X. (6. Def. 1.)

Take OA $= c$, OB (in the opposite direction) $= b$.

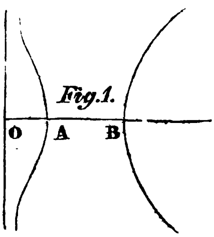
When x is greater than c, or when x is a greater negative quantity than $-b$, y is impossible; hence A and B are the limits of the curve.

When $x = 0$, $y = 0$; when $x = -b$, $y = 0$; and when $x = c$, $y = \infty$; hence the curve must be of the form of the lower conchoid unless it has singular points, to ascertain which we must have recourse to the calculus.

Ex. 3. $y = \pm \sqrt{\dfrac{a.(x-b)\ (x-c)}{x}}$.

This curve is symmetrical on the axis X.

Take OA $= b$, OB $= c$; and the curve will consist of a conchoidal arc passing through A whose base is an ordinate passing through O; and of two infinite branches which meet in B.



If the sign of b is negative, the conchoidal part of the curve is in the 2d and 4th quadrants.

If $b = c$, A and B coincide, and the curve is of the form fig. 2.

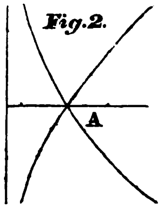


Fig. 2.

If b and c are both impossible quantities, the curve never meets the axis, and its form is that of fig. 3.

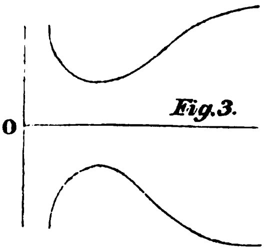
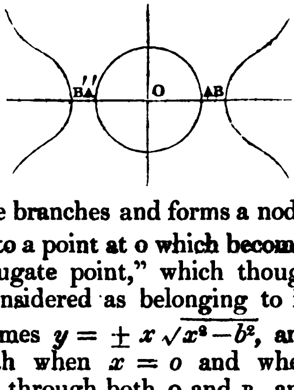


Fig. 3.

Ex. 4. $y = \pm \sqrt{x^4 - (a^2 + b^2)x^2 + a^2b^2}$.

The curve is symmetrical on both the axes.

Take OA $=$ OA$'$ $= a$, a being less than b; and OB $=$ OB$'$ $= b$: the curve shall consist of an oval included between A and A$'$, and of four infinite branches issuing from B and B$'$.



If $a = b$, the oval touches the branches and forms a node.

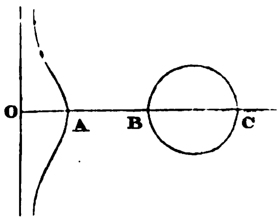
If $a = o$, the oval is reduced to a point at O which becomes what is denominated "a conjugate point," which though detached from the curve is considered as belonging to it. The equation in this case becomes $y = \pm x\sqrt{x^2 - b^2}$, and consequently gives $y = o$ both when $x = o$ and when $x = b$; hence the curve passes through both O and B, and yet at O, or rather in the neighbourhood of O, the curve is impossible.

Def. Conjugate points are such points as are comprised in the equation, but which are separated from the curve.

They always arise as in this example from certain finite portions of the curve vanishing in consequence of assigning particular values to one or more of the constants.

Ex. 5. $y = \pm\sqrt{\dfrac{(a-x)(x-b)(x-c)}{x}}$.

Suppose a, b, c, to be real quantities and in the order of their magnitudes.



Take $OA = a$, $OB = b$, $OC = c$, and the curve is of the form of the annexed figure.

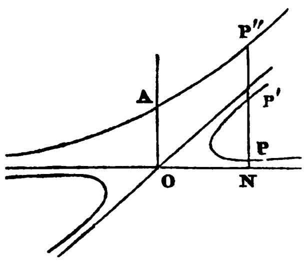
If $b = c$, the oval BC becomes a conjugate point.

If $a = b$, A and B unite, and the curve is of the form of the lower conchoid.

If $a = b = c$, the curve becomes the Cissoid of Diocles. whose equation is $y^2 = \dfrac{(a-x)^3}{x}$.

Ex. 6. $y^3 - 2xy^2 + x^2y - a^3 = 0$.

Suppose the values of y all possible.

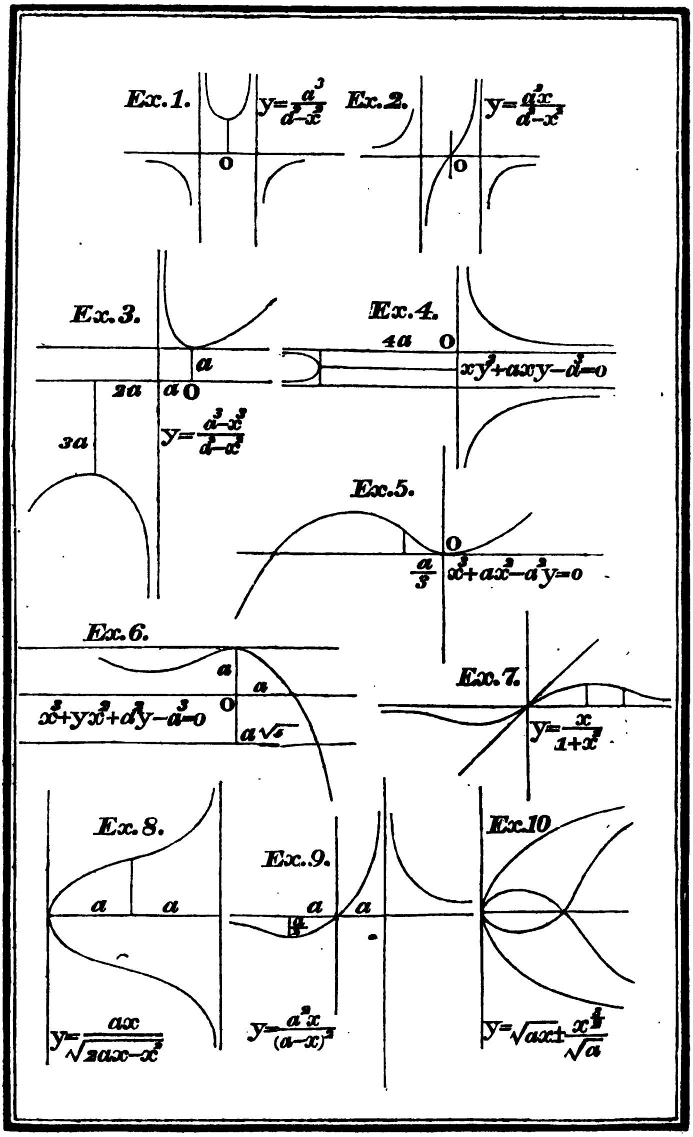


It appears from the changes of the signs that when x is positive, the three values of y are positive, and that when x is negative one value of y is positive and two values negative. (Alg. 311.)

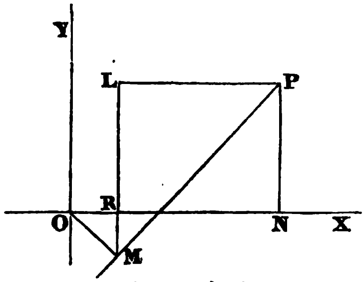
Also, by the solution of a quadratick, $x = y \pm \dfrac{a^{\frac{3}{2}}}{y^{\frac{1}{2}}}$; $\therefore$ when $x = 0$, $y = a$; when $x = \infty$, y may be either 0 or ∞.

Draw then the ordinate $OA = a$, and the form of the curve is that of the annexed figure.

The position and number of the infinite branches will be investigated in the following chapter.


Ex. 1.
y = a^3 / (a^2 − x^2)
Ex. 2.
y = a^2x / (a^2 − x^2)
Ex. 3.
a
2a
a O
3a
y = (a^3 − x^3) / (a^2 − x^2)
Ex. 4.
4a O
xy^2 + axy − a^3 = 0
Ex. 5.
O
a/3
x^3 + ax^2 − a^2y = 0
Ex. 6.
a
a
O
x^3 + yx^2 + a^2y − a^3 = 0
a√5
Ex. 7.
y = x / (1 + x^2)
Ex. 8.
a
a
y = ax / √(2ax − x^2)
Ex. 9.
a
a
y = a^2x / (a − x)^2
Ex. 10
y = √(ax) ± x^{3/2} / √a

18. *Given a curve's equation between one pair of rectangular co-ordinates; required its equation between another pair which have the same origin, and whose position with respect to the first pair is known.*



Let ON, NP; and OM, MP, be the two pairs of co-ordinates.

$ON = x$, $OM = v$
$NP = y$, $MP = w$

Suppose that $\angle$ MON $= \theta$; draw PL, ML, respectively parallel to ON and NP; and let ML cut ON in R.

Then ON = OR + RN or $x = v\cos.\theta + w\sin.\theta$
PN = LM − RM or $y = w\cos.\theta - v\sin.\theta$

which, substituted for x and y in the original equation, will give the required equation between x and y.

In this figure M is supposed to be in the 4th quadrant; if it is in the first, the values of x and y become . . . . . .
$x = v\cos.\theta - w\sin.\theta$
$y = w\cos.\theta + v\sin.\theta$

19. *Required to transfer the origin to a known point, the* direction *of the axes* X *and* Y *remaining unaltered.*

Let A be the new position of the origin, and let the coordinates of O when referred to A be α and β; let x' and y' be the new co-ordinates of P; then $x' = \alpha + x$, $y' = \beta + y$; or $x = x' - \alpha$, and $y = y' - \beta$, which, substituted in the original equation, will give the curve's equation when the origin is transferred to A.

Cor. By means of this and of the preceding article, the equation may be transferred from an axis to any other given axis, the co-ordinates continuing to be rectangular.

Ex. 1. The equation of a curve is $xy = \frac{a^2}{2}$; required to find its equation when the axis X inclines through 45° of the first quadrant.

Substituting as in Art. 18, we have $(v\cos.\theta - w\sin.\theta)(w\cos.\theta + v\sin.\theta) = \frac{a^2}{2}$, or $(\cos.^2\theta - \sin.^2\theta)\,vw + \frac{v^2 - w^2}{2}\sin.2\theta = \frac{a^2}{2}$; and $\sin.\theta = 45°$, $\therefore v^2 - w^2 = a^2$, which is the equation to the axis of a rectangular hyperbola.

Ex. 2. The equation of a curve is $(x^2+y^2)^2 = 2a^2xy$; required its equation when the axis x inclines through 45° of the first quadrant.

By substitution $\{(v\cos.\theta - w\sin.\theta)^2 + (w\cos.\theta + v\sin.\theta)^2\}^2$ $= 2a^2(v\cos.\theta - w\sin.\theta)(w\cos.\theta + v\sin.\theta)$, or $(v^2+w^2)^2$ $= 2a^2((\cos.^2\theta - \sin.\theta)vw + \frac{v^2-w^2}{2}\sin.^2\theta) = a^2(v^2 - w^2)$ when $\theta = 45°$.

Ex. 3. Required to find the general equation of the ellipse.

The equation between the rectangular co-ordinates when the origin is at the centre is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Suppose that the axis major is inclined to the axis x at an $\angle\theta$, and that the co-ordinates of its centre reckoned upon the axes x and Y are α and β.

First, to change the equation to axes parallel to x and Y, we have $\frac{1}{a^2}(v\cos.\theta + w\sin.\theta)^2 + \frac{1}{b^2}(w\cos.\theta - v\sin.\theta)^2 = 1$, or $\left(\frac{1}{a^2} - \frac{1}{b^2}\right)vw.\sin.2\theta + \left(\frac{v^2}{a^2} + \frac{w^2}{b^2}\right)\cos.^2\theta + \left(\frac{v^2}{b^2} + \frac{w^2}{a^2}\right)$ $\sin.^2\theta = 1$, in which, if we substitute for v and w, $v' - \alpha$ and $w' - \beta$, and change v' and w' into x and y, there will result the general equation of the ellipse. It is of the form $ay^2 + (b + cx)y + e + fx + gx^2 = 0$.

Since the equation of the ellipse or hyperbola contains *five* arbitrary constants, viz. α and β, which denote the position of its centre, θ the direction, and a and b the magnitudes of the semi-axes; it may be drawn fulfilling five conditions: it may be made for instance to pass through five points.

20. If the variables are raised to *even* powers in each term of the equation, or if the equation is such that a variable enters into every term, and that the sum of the exponents in each term is *odd*; in either case the origin is the centre of the curve. (Art. 6. Cor. Def. 2.) Thus, the origin is the centre of $y^4 - 2x^2y^2 + x^4 - a^2x^2 + b^2 = 0$, and also of $y = \frac{x}{1+x^2}$, or its equal $y + x^2y - x = 0$. (Praxis 17. Examples 2 and 7.)

But if the origin is not the centre; in order to ascertain whether the curve has a centre, we must substitute for x

and y, $x + \alpha$ and $y + \beta$, where α and β are indeterminate quantities; and determine from the resulting equation whether there are any values of α and β, which cause the terms containing the odd powers to vanish, if the equation is of an even degree, or those containing the even powers to vanish if it is of an odd degree.

Ex. To determine whether curves have a centre whose equation is of the second degree.

The general equation is $ay^2 + (b + cx)y + c + fx + gx^2 = 0$; in which substitute $y + \beta$ and $x + \alpha$ for y and x; and in order that the curve may have a centre, the condition is $(2a\beta + b + c\alpha)\,y + (c\beta + f + 2g\alpha)\,x = 0$; consequently the conditions are $c\alpha + 2a\beta + b = 0$, and $2g\alpha + c\beta + f = 0$; whence, by elimination, $\alpha = \dfrac{cb - 2af}{4ag - c^2}$, and $\beta = \dfrac{cf - 2bg}{4ag - c^2}$, which are finite except when $c^2 = 4ag$, or when the curve is a parabola. (Alg. 508.)

Similar curves.

21. *Def.* Two curves are said to be *similar* if there is a ratio in which any abscissæ being taken their corresponding ordinates are in the same ratio.

Ex. If in two circles abscissæ be taken in the ratio of their diameters, it may be shown that the ordinates are in the same ratio; or circles are similar curves.

When the ordinates are thus drawn, they mark the corresponding points of the two curves; and lines which are drawn joining corresponding points are said to be *similarly situated.*

22. *Algebraick curves are similar, the equations of which are homogeneous, and which contain but one parameter.*

Let a be the parameter, and let the equation be of n dimensions; dividing each term by a^n it will be seen that $\dfrac{y}{a} = \mathrm{F}\dfrac{x}{a}$.

Let a' be the parameter of another curve whose equation is of the same form as the first; and take $x' : x :: a' : a$, and let y' be the ordinate corresponding to x'; then we have $\dfrac{y'}{a'} = \mathrm{F}\dfrac{x'}{a'} = \mathrm{F}\dfrac{x}{a} = \dfrac{y}{a}$; therefore $y' : y :: a' : a$, or the curves are similar.

Hence circles, all parabolas of the same order, rectangular hyperbolas, Cissoids of Diocles, Lemniscatas are similar curves.

Cor. If the curve is a transcendental, and its equation is included under the form $\frac{y}{a} = \text{F}\,\frac{x}{a}$, all such curves are similar.

Thus, catenaries are similar; for the equation is $\frac{2y}{a} = e^{\frac{x}{a}} + e^{-\frac{x}{a}}$.

23. *Algebraick curves of more than one parameter are similar when their equations are homogeneous, and their corresponding parameters are to each other in the same ratio.*

Let $a, b, \ldots.$ and $a', b', \ldots.$ be the parameters of two curves $u = 0$ and $u' = 0$, where the equation is homogeneous and of n dimensions.

It will be shown (Vol. 2. Ch. 2.) that $u = a^n \text{F}\left(\frac{y}{a}, \frac{x}{a}, \frac{b}{a} \ldots\right)$; consequently, we have $\text{F}\left(\frac{y}{a}, \frac{x}{a}, \frac{b}{a} \ldots\right) = 0$. Similarly, taking in the other curve, $x' : x :: a' : a :: b' : b$, and y' being the ordinate of x', it may be shown that $\text{F}\left(\frac{y'}{a'}, \frac{x'}{a'}, \frac{b'}{a'} \ldots\right) = 0 = \text{F}\left(\frac{y}{a}, \frac{x}{a}, \frac{b}{a} \ldots\right)$; in which $\frac{x}{a} = \frac{x'}{a'}$, $\frac{b}{a} = \frac{b'}{a'}$, &c. = &c.; whence it follows that $\frac{y'}{a'} = \frac{y}{a}$, & $y' : y :: a' : a$, or the curves are similar.

Ex. Ellipses whose axes are in the same ratio are similar. Conchoids of Nicomedes are similar when the ratio of the modulus to the distance of the node from the base is the same in the two curves.

24. *When transcendentals which contain but one parameter* (a) *are similar, their equation is such that* $\frac{\text{dy}}{\text{dx}} = \text{F}\left(\frac{\text{y}}{\text{a}}, \frac{\text{x}}{\text{a}}\right)$.

Suppose that x flows uniformly, and always takes in the other curve as before $x' : x :: a' : a$, and let y' be the ordinate of x'.

Since the curves are similar, inc. x' : inc. x :: inc. y' : inc. y :: $a' : a$; and taking these ratios in their limit, $dx' : dx$ :: $dy' : dy$, or $\frac{dy}{dx} = \frac{dy'}{dx'}$; hence, the form of $\frac{dy}{dx}$ must be such that its value can receive no change by substituting y', x', a' for y, x, and a respectively, or $\frac{dy}{dx} = \mathrm{F}\left(\frac{y}{a}, \frac{x}{a}\right)$.

25. *Conversely, if the equation is such that* $\frac{dy}{dx} = \mathrm{F}\left(\frac{y}{a}, \frac{x}{a}\right)$, *the curves are similar.*

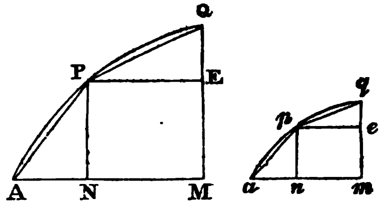
For, taking x' and y' as before, since the form of the equation is given, we shall have $\frac{dy'}{dx'} = \mathrm{F}\left(\frac{y'}{a'}, \frac{x'}{a'}\right) =$, by the construction, $\mathrm{F}\left(\frac{y'}{a'}, \frac{x}{a}\right)$, from which and the proposed equation, we have $\frac{x}{a} = f\left(\frac{dy'}{dx'}, \frac{y'}{a'}\right) = f\left(\frac{dy}{dx}, \frac{y}{a}\right)$, whence it follows that $\frac{dy'}{dx'} = \frac{dy}{dx}$, and $\frac{y'}{a'} = \frac{y}{a}$, or $y' : y :: a' : a ::$ $x' : x$, which is the characteristick property of similar curves.

Thus catenaries, cycloids, involutes of circles, are all similar curves.

Cor. When the equation contains more than one parameter, a, b, ...; the curves are similar if $\frac{dy}{dx} = \mathrm{F}\left(\frac{y}{a}, \frac{x}{a}, \frac{b}{a}, \ldots\right)$.

26. *Similarly situated lines are equally inclined to each other and to the axes of the abscissæ, and are to each other in a constant ratio, viz.—that of the parameters of the curves.*

For, let AP, ap, be corresponding parts of similar curves; draw the co-ordinates AN, AP; an, np; join AP, ap.



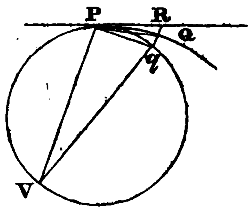
Then, since AN : NP :: an : np, and that ∠ ANP = ∠ anp, therefore (Eu. 6. 7.) the triangles ANP, anp, are similar, and the ∠ PAN = the ∠ pan, and AP : ap :: AN : an :: NP : np :: the parameter of APQ : the parameter of apq.

In the same manner it may be shown, if Q and q are any other corresponding points, and PQ, pq be joined, that PQ : pq :: AP : ap, and that PQ, pq are equally inclined to AP, ap, or to AN, an.

Cor. Conversely, lines which make equal angles with any similarly situated lines are themselves similarly situated.

27. Lemma. *If the arc of a curve be gradually diminished and at length vanish, the angle contained between its chord and tangent shall ultimately vanish.*

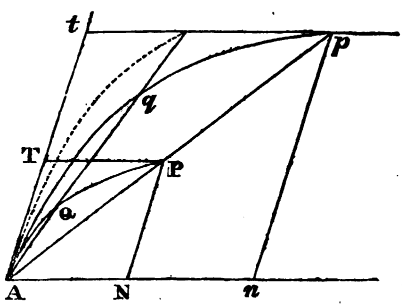
Let PQ be the arc; Q moving towards P, always draw QR making a finite angle with the tangent PR. Let PVq be any circle touching PR and falling within the curve PQ; produce RQ to meet it in q; and join PQ, Pq, PV, Vq.



Then, RQq moving towards P, the arc Pq or the ∠ PVq, and therefore its equal, the ∠ RPq, and à fortiori the ∠ RPQ gradually diminishes and at length vanishes.

28. *Similar and conterminous arcs have a common tangent at the point where they meet.*

Let AP, Ap, be any corresponding parts of similar curves which have the same diameter AN; draw the chords AP, Ap; which (Art. 26.) coincide.



Now suppose APp to revolve round A so that the chords AP, Ap, and consequently the arcs may gradually dimi-

nish and at length vanish; then, since AP : Ap is a constant ratio, they vanish at the same time; and their ultimate direction (27. Lemma) is that of their respective tangents at A; or the tangents at A coincide.

29. *While an arc is diminishing there may be always drawn a finite arc similar to it and having a common chord and tangent.*

Let AP be any magnitude of an arc whose tangent is AT; draw PT parallel to the axis, making a finite angle with the tangent. In AT, take At of finite and *constant* magnitude; draw tp parallel to TP, meeting AP produced in p. Let AQ be any magnitude of the same arc less than AP; join AQ and produce it to q, making Aq : AQ :: Ap : AP :: At : AT, then q shall trace an arc similar to AP.

For, drawing the co-ordinates of Q and q, they may be shown to be in the same ratio with those of P and p, which being constant, it follows from the definition (21.), that Aqp is similar to AQP.

The curves have a common tangent at A by the preceding article.

As P moves towards A, t being fixed, the ratio At : AT gradually increases and approximates to infinity as its limit; consequently, for every new value of the arc, a different curve Aqp will be constructed: it is obvious that its curvature at all points diminishes, and that its limit is that of a right line.

30. *If from one extremity of an arc a line be always drawn making a finite angle with the tangent at the other extremity, and the arc be indefinitely diminished and at length vanish; the ultimate ratio of the arc : the chord : the tangent is a ratio of equality.*

For (fig. 28.), let AP be the arc; draw PT making a finite angle with the tangent AT; and, as in the preceding article, P moving towards A, always describe a curve Ap similar to AP having a common chord and tangent; then pt is parallel to PT.

By a well known property of similar figures AP : AQP : AT :: Ap : Aqp : At; and, to find the ultimate ratio of these finite lines, let P move towards A, then the $\angle$ pAt gradually diminishes and at length vanishes (27. Lemma), and the $\angle$ Atp is finite; consequently the angles Atp, Apt approximate to two right angles as their limit, and their sines are ultimately equal, and the limiting ratio of Ap : At (Trig. p. 27),

and à fortiori the limiting ratio of Ap : Aqp : At, which is the same as that of AP : AQP : AT, is a ratio of equality. *Principia, vol.* 1, *Lem.* 7.

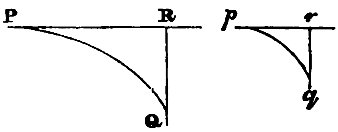
31. *Curvature.*

Def. 1. The ratio of the *subtenses* of two arcs is the limiting ratio of two lines drawn from their extremities, making *finite* angles with their tangents at the other extremities; the arcs being gradually diminished, and vanishing at the same time.

The term subtense, like that of fluxion, is essentially relative; and yet, for the sake of convenience, it is frequently considered as expressing absolute magnitude.

By varying the angle which the subtense makes with the tangent, the same arc may have an infinite number of subtenses; they are in the ratio of 1 : 2, if the one is perpendicular to the tangent and the other inclined at ∠ 30°.

32. *Def.* 2. *Curvature* is measured by the perpendicular subtense of a given arc.



Thus, to compare the curvature at any two points P and p; take PQ, pq, small and *equal* arcs; draw QR, qr, perpendicular to the tangents; then, diminishing PQ and pq indefinitely, the curvature at P : the curvature at p = the limit of QR : qr.

33. *The curvature at different points of the same circle is invariable, and the curvature of different circles varies inversely as their radii.*

Take PQ any small arc of the circle, draw QR perpendicular on the tangent at P; draw the diameter PCV, which is parallel to QR, C being the centre of the circle, and join PQ, QV.

From similar triangles we have $\text{QR} = \dfrac{\text{PQ}^2}{\text{PV}} = \dfrac{\text{PQ}^2}{2\text{CP}}$, which in the limit $= \dfrac{\text{arc}^2\text{PQ}}{2\text{CP}}$; therefore the curvature, which varies as the limit of QR when PQ is given, is invariable in the same circle, and varies inversely as the radius in different circles.

Cor. 1. When the radius vanishes the curvature becomes

infinite; and when the radius becomes infinite the curvature vanishes, or its limit is that of a right line.

Cor. 2. By varying the radius of a circle we may have all possible degrees of curvature, from nothing to infinity.

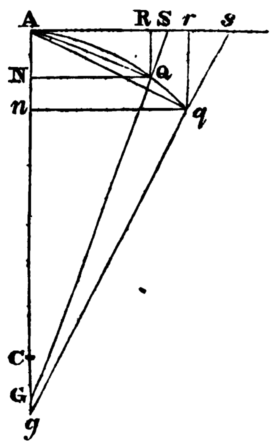
It is on account of this property of circles that they are taken to measure the curvature of all curves of *finite* curvature.

34. *Def.* 3. The *circle of curvature* to any point of a curve, is that circle which has the same curvature which the curve has at that point.

The *chord of curvature* is a chord of the circle of curvature, drawn parallel to the subtense of the arc, making a finite angle with the tangent.

Cor. Hence, if the perpendicular line RQ be produced to meet the circle of curvature in q (vid. fig. 27.), the limiting ratio of RQ : Rq is a ratio of equality.

35. *The subtenses of conterminous arcs are to each other in the duplicate ratio of their arcs.*



Case 1. Let the subtenses be *perpendicular* to the tangent.

Let QR, qr, in the limit, be the perpendicular subtenses of two conterminous arcs, AQ, Aq, which are gradually diminished, and which vanish together.

Draw AGg at right angles to AR; join AQ, Aq; and draw QG, qg, at right angles to AQ, Aq, meeting AGg in G and g.

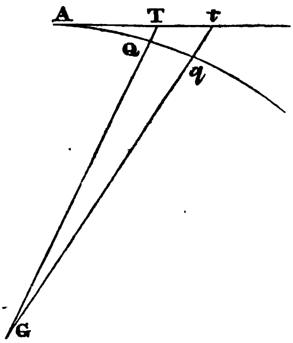
Then, since the angles AQG, Aqg, are right angles, and that QAG, qAG, are ultimately right angles (27. Lemma); therefore QG, qg, ultimately intersect AG at a finite distance from A; let C be the limit of their intersections.

From similar triangles $\mathrm{QR} = \frac{\mathrm{AQ}^2}{\mathrm{AG}}$, $qr = \frac{\mathrm{A}q^2}{\mathrm{A}g}$; therefore the subtense of AQ : subtense of Aq, which = the limit of QR : qr, = the limit of $\frac{\mathrm{AQ}^2}{\mathrm{AG}} : \frac{\mathrm{A}q^2}{\mathrm{A}g} = \frac{\mathrm{arc}^2\mathrm{AQ}}{\mathrm{AC}} : \frac{\mathrm{arc}^2\mathrm{A}q}{\mathrm{AC}} =$ $\mathrm{arc}^2\mathrm{AQ} : \mathrm{arc}^2\mathrm{A}q$.

Case 2. Next, let the subtenses make any the same angle with the tangent.

Let QS, qs, be the lines which in the limit become the subtenses, and drawing QR, qr perpendicular on the tangent; the subtense of AQ : the subtense of Aq = the limit of QS : qs = (from similar triangles) the limit of QR : qr = arc^{2}AQ : arc^{2}Aq.

Case 3. Suppose that the lines TQ, tq are not parallel.



Produce TQ, tq, to meet in G; then, since AQ, Aq vanish together, the $\angle$ G gradually diminishes and ultimately vanishes; and the $\angle^s$ T and t are finite, consequently in the limit, TQ and tq are parallel, and therefore by the former case, the subtenses are in the duplicate ratio of their arcs. (Principia, vol. 1, Lem. 11.)

Cor. If AC be drawn parallel to QR, and C is the limit of the intersections of ACG and QG, AC is the diameter of the circle of curvature to the point A or its chord, according as the subtense is perpendicular or acute.

36. *Any arc of finite curvature which is gradually diminished and ultimately vanishes, possesses in the limit the properties of a parabolick arc.*

For, the construction in fig. 35 remaining, draw QN, qn, parallel to AR, or perpendicular on AC; then AN = RQ and An = rq; therefore in the limit AN : An :: AQ^2 : Aq^2 :: (30) AR^2 : Ar^2 :: NQ^2 : nq^2, which is the property of an Apollonian parabola; or AQ in the limit may be considered as the arc of a parabola whose axis is ANG and parameter AC, the diameter of the circle of curvature.

If the subtense is inclined to the tangent at an acute angle, ANG is a diameter of the parabola, and AC, which is now the chord of curvature, is a latus rectum of the parabola.

Ex. Thus the equation of a circle is $y^2 = ax - x^2$, which in the limit, takes the form of $y^2 = ax$, the equation of a parabola whose parameter $= a$.

Cor. Hence, finite curvature may also be measured by the curvature at the vertex of an Apollonian parabola, and it varies inversely as the latus rectum of the parabola.

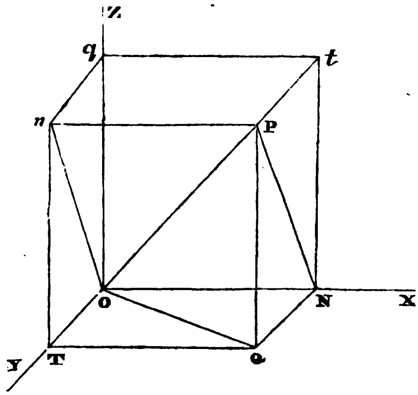
37. Curvature varies as the perpendicular subtense directly and the duplicate ratio of the arc inversely.

For when the arc is given, the perpendicular subtense varies as the curvature (32 Def.); and when the curvature is given, the subtense varies in the duplicate ratio of the arc (35); therefore, when neither is given, the subtense varies as the curvature and the duplicate ratio of the arc conjointly; or curvature varies as the perpendicular subtense directly and the duplicate ratio of the arc inversely.

38. *Points in Space.*

If the plane in which the point is situated is not given, the position of the point is determined by its distance from three planes, each of which is at right angles to the other two. These distances are called its *rectangular co-ordinates.*

Let XOY, XOZ, ZOY, be the three planes, each of which is at right angles to the other two; O the origin of the co-ordinates, P the proposed point; draw PQ perpendicular to the plane XOY, and in this plane draw QN perpendicular to OX; then the distances of P from the planes ZOY, ZOX, YOX, are ON, NQ, QP, which are therefore its rectangular co-ordinates.



Let ON $= x$, NQ $= y$, QP $= z$; then the position of P in fixed space evidently depends upon the values of x, y, and z.

39. There are eight solid angles which may be described round O, and consequently we may have eight different positions corresponding to the same magnitude of x, y, and z; but the ambiguity will be removed by attending to their signs.

Suppose that XOY is in the plane of the paper, P being above the plane; then the signs of P's co-ordinates in the

four quadrants are $\left.\begin{array}{c}+++\\-++\\--+\\+-+\end{array}\right\}$; and when P is below, the signs of its co-ordinates are $\left.\begin{array}{c}++-\\-+-\\---\\+--\end{array}\right\}$, all which are different.

In the diagram of the preceding article P is situated in the fourth upper quadrant.

40. *The sum of the squares of the co-ordinates of a point is equal to the square of its distance from the origin.*

For, the same construction remaining, complete the parallelepiped ONQP, and join OP, OQ.

Then PQ is at right angles to OQ (Eu. 11. Def. 3.), and therefore (Eu. 1. 47.) $\text{OP}^2 = \text{OQ}^2 + \text{QP}^2 = \text{ON}^2 + \text{NQ}^2 + \text{QP}^2$.

If $\text{OP} = r$, $\text{ON} = x$, $\text{NQ} = y$, $\text{QP} = z$; then $r = \sqrt{x^2 + y^2 + z^2}$.

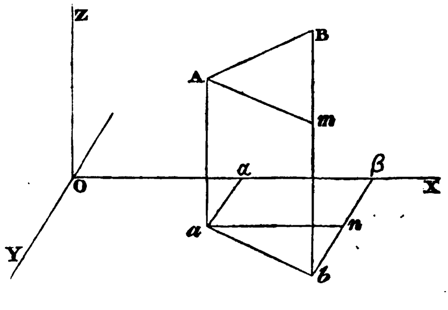
Cor. 1. The square of a right line passing through the origin is equal to the sum of the squares of its projections on the three rectangular axes.

For, joining PN, the $\angle$ ONP is a right angle (Eu. 11. Def. 3.), or ON is the projection of OP on the axis X; similarly NQ and QP are equal to its projections on the axes Y and Z.

Cor. 2. If v, v', v'' are the angles which OP makes with the axes X, Y and Z respectively; $\cos. v = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}$,

$\cos. v' = \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}$, and $\cos. v'' = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$.

41. *Required the distance of two given points in space.*



Let A and B be the points; x, y, z, and x', y', z', their co-ordinates; draw the co-ordinates as in the figure; also draw Am perpendicular to Bb, or parallel to ab, and an parallel to OX, or perpendicular to $b\beta$.

(Eu. 1. 47.) $AB^2 = Am^2 + Bm^2 = ab^2 + Bm^2 = an^2 + nb^2 + Bm^2 = \alpha\beta^2 + nb^2 + Bm^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2$, or $AB = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}$.

Cor. If A coincides with O, $OB = \sqrt{x^2 + y^2 + z^2}$.

42. *The sum of the squares of the cosines of the angles, which a line drawn from the origin makes with the three rectangular axes, is equal to the square of radius.*

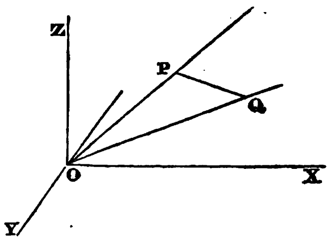
(Fig. 38.) Let v, v', v'' be the angles which OP makes with the axes X, Y, Z respectively; then, (40. Cor. 2.)

$\cos. v = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}$, $\cos. v' = \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}$, $\cos. v'' = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$, wherefore $\cos.^2 v + \cos.^2 v' + \cos.^2 v'' = 1 = (\text{rad.})^2$.

Cor. 1. $\sin.^2 v + \sin.^2 v' + \sin.^2 v'' = 2$.

Cor. 2. If two of the angles be given, the third is determined.

43. *Required to find the angle contained between two right lines which are drawn from the origin, inclined at given angles to the axes.*



Let OP, OQ be the lines inclined to the axes X, Y, Z at angles v, v', v'' and w, w', w'' respectively; let x, y, z and x', y', z' be the co-ordinates of P and Q, any points in the lines; let $OP = r$, $OQ = r'$; join PQ.

Then (Art. 41.), $PQ^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2$; but (Trig. p. 24) $PQ^2 = r^2 + r'^2 - 2rr' \cos. POQ = x^2 + y^2 + z^2 + x'^2 + y'^2 + z'^2 - 2rr' \cos. POQ$; therefore $\cos. POQ = \dfrac{xx'}{rr'} + \dfrac{yy'}{rr'} + \dfrac{zz'}{rr'} = \cos. v \cos. w + \cos. v' \cos. w' + \cos. v'' \cos. w''$; from which formula $\angle$ POQ may be computed.

Cor. When $\angle POQ = 90°$, $\cos. v \cos. w + \cos. v' \cos. w' + \cos. v'' \cos. w'' = 0$.

44. *Required to change the direction and position of the rectangular axes.*

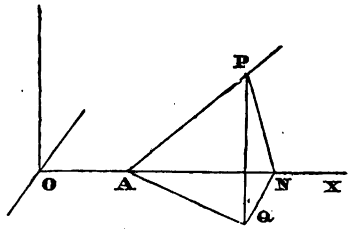
Their *direction* is changed by means of the preceding article; for the inclination of OP to the original axes is given, and also the angles which the original axes make with the new axes; and consequently the inclination of OP to the new axes may be calculated.

When the direction has been changed, to change their position, the co-ordinates of the new origin must be added to or subtracted from those of the original.

For the formulæ for changing the inclination of the co-ordinates, vid. Lacroix, 1. 4. 182.

45. *Def.* The equation of a surface is that which expresses the relation between the co-ordinates of any point in it.

Ex. Required the equation of a right line intersecting one of the rectangular planes at a given point of the axis and at a given angle.



Let AP cut the axis OX in A; take P any point in it, and draw its co-ordinates ON, NQ, QP; join AQ.

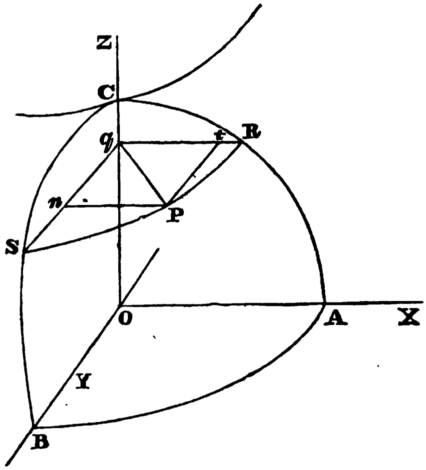
Let OA $= a$, and $\alpha =$ the given angle; then (Eu. 11. Def. 5.) $\angle$ PAQ $= \alpha$.

Also, PQ $=$ AQ $\tan.\alpha = \tan.\alpha\,\sqrt{\text{AN}^2 + \text{NQ}^2}$; *i. e.* $z = \tan.\alpha\,\sqrt{(x-a)^2 + y^2}$, which is the required equation.

This is the equation of a conical surface, the axis coincident with X, its vertical angle α, and the distance of its vertex from the origin $= a$.

46. Solids are frequently characterized by two sections which intersect in the axis at right angles, and by the nature of the variable curve which, moving along the axis perpendicular to it, generates the surface.

If it is a solid of revolution, the variable curve is a circle.



Let AOC, BOC be two curves in the planes XZ, YZ, having a common axis OC; and let a plane SPRq move along CO perpendicular to it; then, if the nature of the curve SPR be given, the equation of the surface which it generates can be found.

For, taking any point P in the curve, draw Pn perpendicular to Sq; and complete the rectangular parallelogram Pnqt; and P's co-ordinates are equal to Oq, qn, nP; but the sides of the rectangular parallelogram are the co-ordinates of SPR on the axes Rq, Sq, and are therefore known functions of Rq, Sq, and consequently of Oq their common ordinate.

In the following examples we suppose that . . . .
OA $= a$, R$q = v$
OB $= b$, S$q = w$
OC $= c$.

Cor. Conversely, if the equation of the surface be known, its general form and outline may be traced.

For the co-ordinate sections may be determined from the equation, and all sections parallel to them which will give the form of the solid. Thus, if we have $\frac{x^2}{a} + \frac{y^2}{b} = z$; the co-ordinate sections in the planes XZ and YZ are parabolas whose equations are $x^2 = az$ and $y^2 = bz$: and every section parallel to XY has its equation $\frac{x^2}{az} + \frac{y^2}{bz} = 1$, which, since z is constant, is an ellipse.

Ex. 1. Let CA, CB be right lines and also SPR a right line; required the equation of the surface.

$$\text{Here } \frac{x}{v} + \frac{y}{w} = 1\text{; but since } \left.\begin{array}{r} \frac{z}{c} + \frac{v}{a} = 1 \\ \text{and } \frac{z}{c} + \frac{w}{b} = 1 \end{array}\right\}\text{, therefore}$$

$\frac{v}{a} = \frac{w}{b}$; whence, multiplying the first equation by $\frac{v}{a}$ or its equal $\frac{w}{b}$, there results $\frac{x}{a} + \frac{y}{b} = \frac{v}{a} = 1 - \frac{z}{c}$; or $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ is the required equation.

This is one of the equations of a plane surface.

Ex. 2. Required the equation of a *sphere*.

Here CA, CB, SPR are circles; therefore we have $\frac{x^2}{v^2} + \frac{y^2}{v^2} = 1$; but $v^2 = a^2 - z^2$, or the equation is $x^2 + y^2 + z^2 = a^2$.

If the origin is not at the centre, the equation is $(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = a^2$, where α, β, γ are the co-ordinates of the centre.

This equation containing four arbitrary constants, a sphere may be described fulfilling four conditions; it may be drawn, for instance, through four points, or touching four planes.

If it be required to describe a sphere touching the three co-ordinate planes; the co-ordinates of its centre are each equal to the radius, and the equation becomes $(x-a)^2 + (y-a)^2 + (z-a)^2 = a^2$, or $x^2 + y^2 + z^2 - 2a(x+y+z-a) = 0$; and as this contains one indeterminate quantity, the sphere may be described fulfilling a fourth condition; it may be made to pass through a given point, or to touch a given sphere or a given plane.

Ex. 3. Let CA be a circle, but the section CB an ellipse; also suppose SPR a right line.

Here $\frac{x}{v} + \frac{y}{w} = 1$; but $v = \sqrt{a^2 - z^2}$, & $w = \frac{b}{a}\sqrt{a^2 - z^2}$, therefore the required equation is $\frac{x}{a} + \frac{y}{b} = \frac{\sqrt{a^2 - z^2}}{a}$.

Ex. 4. In the preceding example, suppose that the locus of S is a right line parallel to OC.

Here $w = b$ and the equation is $\frac{x}{\sqrt{a^2 - z^2}} + \frac{y}{b} = 1$.

This is the *convex wedge*.

If in *Ex.* 2. the generating surface S*q*RV is a square, the

solid is the fourth part of a *groin*. There are elliptick and parabolick groins; also, the generating surface may be supposed to be an oblong, in which case the surfaces of two contiguous sides will not be similar and equal.

Ex. 5. Required the equation of a *spheroid*.

Def. A spheroid is the solid generated by the revolution of an ellipse round one of its axes.

Here CA, CB are equal ellipses, and SPR is a circle; hence we have $\frac{x^2}{v^2} + \frac{y^2}{v^2} = 1$, and $\frac{v^2}{a^2} + \frac{z^2}{b^2} = 1$; therefore the equation is $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$.

If the origin is not at the centre, the equation becomes $\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{a^2} + \frac{(z-\gamma)^2}{b^2} = 1$; which shows that a spheroid may be drawn fulfilling five conditions.

Ex. 6. Required the equation of an *ellipsoid*.

Def. This is not a solid of revolution: CRA, CSB are ellipses having a common semiaxis OC; RPS an ellipse whose semiaxes are Rq, Sq.

Here $\frac{x^2}{v^2} + \frac{y^2}{w^2} = 1$; also, since $\left.\begin{array}{r} \frac{z^2}{c^2} + \frac{v^2}{a^2} = 1 \\ \text{and } \frac{z^2}{c^2} + \frac{w^2}{b^2} = 1 \end{array}\right\}$,

therefore $\frac{v^2}{a^2} = \frac{w^2}{b^2}$; and, multiplying the first equation by $\frac{v^2}{a^2}$, or its equal $\frac{w^2}{b^2}$, there results $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{v^2}{a^2} = 1 - \frac{z^2}{c^2}$; or the equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Ex. 7. Required the equation of the common *paraboloid*.

Take the vertex C for the origin; and let a = the latus rectum.

Here $x^2 + y^2 = \text{P}q^2 = \text{R}q^2 = az$.

Ex. 8. Required the equation of the *elliptick paraboloid*.

Here CA, CB, are parabolas with different latera recta, and SPR is an ellipse whose semiaxes are the ordinates of the parabolas.

Let a and b be the latera recta of CA and CB; then we have $\frac{x^2}{v^2} + \frac{y^2}{w^2} = 1$; but $v^2 = az$ and $w^2 = bz$, therefore the equation is $\frac{x^2}{a} + \frac{y^2}{b} = z$.

Cor. z cannot be negative, or no part of the solid lies below XY.

Ex. 9. Required the equation of the common *hyperboloid.*

Let OC $= c =$ the semiaxis of the hyperbola, and OA $=$ OB $= a =$ the semi-transverse; then, since SPR is a circle $\frac{x^2 + y^2}{v^2} = 1$; also, since z and v are the co-ordinates of an hyperbola, whose semiaxes are c and a, we have $\frac{z^2}{c^2} - \frac{v^2}{a^2} = 1$ (7. 14. Ex. 2.); and the required equation is $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$.

If the axis of the solid coincides with the axis X; then, a being the principal semiaxis of the two hyperbolas, and b their semi-transverse, the equation becomes $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{b^2} = 1$.

Ex. 10. Required the equation of the *elliptick hyperboloid.*

Def. Here CA$'$, CB$'$ are hyperbolas with unequal transverse axes OA, OB; and RPS is an ellipse.

Hence we have $\frac{x^2}{v^2} + \frac{y^2}{w^2} = 1$; also since $\left.\begin{array}{r}\frac{z^2}{c^2} - \frac{v^2}{a^2} = 1 \\ \text{and } \frac{z^2}{c^2} - \frac{w^2}{b^2} = 1\end{array}\right\}$,

therefore $\frac{v^2}{a^2} = \frac{w^2}{b^2}$; and multiplying the first equation by these equal magnitudes, there results $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{v^2}{a^2} = \frac{z^2}{c^2} - 1$, or the equation is $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The equation on the axis x is $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

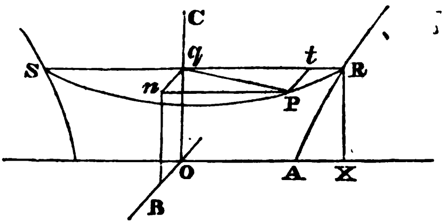
Cor. 1. When $z = 0$, the equation becomes $-\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which is impossible, or the solid does not intersect the plane XY.

Cor. 2. Every section parallel to XY is an ellipse whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2 - c^2}{c^2}$. The sections parallel to XZ and YZ are hyperbolas.

Ex. 11. Required the equation of the *transverse hyperboloid.*

Def. This is generated by the revolution of an hyperbola round its transverse axis; and its surface may be supposed to be generated by a variable circle moving along the transverse axis; the radius of the circle being an abscissa of the hyperbola.

Here $\frac{v^2}{a^2} - \frac{z^2}{b^2} = 1$, and $v^2 = x^2 + y^2$, therefore the equation is $\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 1$.



Ex. 12. Required the equation of the *elliptick transverse hyperboloid.*

Def. Here the principal axes of the hyperbolas in the planes XZ, YZ are unequal, and they have a common transverse axis OC; also SPR is an ellipse, whose semi-axes are the abscissæ of the hyperbolas.

Hence we have $\frac{x^2}{v^2} + \frac{y^2}{w^2} = 1$; but since $\left.\begin{array}{r} \frac{v^2}{a^2} - \frac{z^2}{c^2} = 1 \\ \text{and } \frac{w^2}{b^2} - \frac{z^2}{c^2} = 1 \end{array}\right\}$,

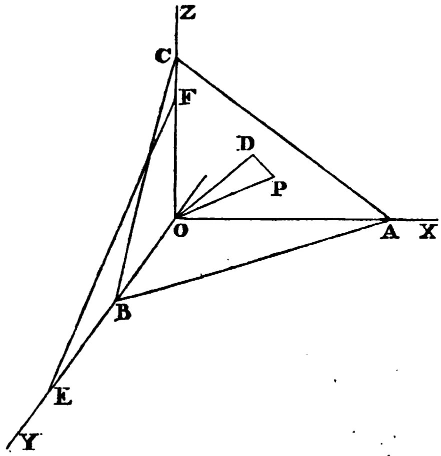
therefore $\frac{v^2}{a^2} = \frac{w^2}{b^2}$, and consequently the required equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

The equation on the axis X is $\frac{y^2}{b^2} + \frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$.

Spheroids and ellipsoids have equations of the same form, whether the axis major of the ellipse or its transverse be the axis of the solid; but the equations of the hyperboloids differ essentially from those of the transverse solids; the former containing two and the latter only one negative sign. The forms of their surfaces are also essentially different: the one consisting of two distinct surfaces formed by the two opposite hyperbolas; the other, of one continued surface.

47. *Required the equation of a plane.*

The position of a plane depends upon the magnitude and position of a line drawn from the origin perpendicular to it. Its characteristick property is, that the angle made by this perpendicular, and the line which joins it with any point in the surface, is a right angle. (Eu. 11. Def. 3.)



Let OD be the perpendicular on the plane, P any point in it, join DP, then $\angle$ ODP is a right angle, and $OP^2 = OD^2 + DP^2$.

Let x, y, z and a, b, c be the coordinates of P and of D; then, by substitution, we have $x^2 + y^2 + z^2 = a^2 + b^2 + c^2 + (x-a)^2 + (y-b)^2 + (z-c)^2$, or $ax + by + cz - (a^2 + b^2 + c^2) = 0$; a simple equation of three variables containing three arbitrary constants.

Cor. A plane may be drawn fulfilling three conditions; it may be drawn for instance through two given points, making a given angle with a given plane.

48. *If* A, B, *and* C, *are the distances from the origin at which a plane intersects the axes* X, Y, *and* Z *respectively; its equation is* $\frac{X}{A} + \frac{Y}{B} + \frac{Z}{C} = 1$.

For, draw OD from the origin perpendicular to the plane, and let a, b, c be the co-ordinates of D; then the plane's equation (47.) is $ax + by + cz - (a^2 + b^2 + c^2) = 0$, or $\frac{ax}{a^2+b^2+c^2} + \frac{by}{a^2+b^2+c^2} + \frac{cz}{a^2+b^2+c^2} = 1$.

Let the plane cut the axes X, Y, and Z in the points A, B, and C respectively, and let OA = A, OB = B, OC = C; join AD.

Since ODA is a right angle (Eu. 11. Def. 3.), therefore

$$\text{OA or A} = \frac{\text{OD}}{\cos. \angle \text{AOD}} = \sqrt{a^2 + b^2 + c^2} \div \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$= \frac{a^2+b^2+c^2}{a}$. Similarly $\text{B} = \frac{a^2+b^2+c^2}{b}$, and $\text{C} = \frac{a^2+b^2+c^2}{c}$; whence, by substitution, the plane's equation becomes $\frac{x}{\text{A}} + \frac{y}{\text{B}} + \frac{z}{\text{C}} = 1$, which is the same equation as that deduced Art. 46. Ex. 1.

Cor. 1. If the plane is parallel to one of the co-ordinate planes as XY, then $\text{A} = \infty$ and $\text{B} = \infty$, and its equation becomes $\frac{z}{\text{C}} = 1$, or $z = \text{C}$.

Cor. 2. If the plane is perpendicular to XY; $\text{C} = \infty$, and the equation becomes $\frac{x}{\text{A}} + \frac{y}{\text{B}} = 1$, or $x = -\frac{\text{A}}{\text{B}}y + \text{A}$.

Cor. 3. Multiplying the equation by C, we have $\frac{\text{C}}{\text{A}}x + \frac{\text{C}}{\text{B}}y + z = \text{C}$, or $z + x \tan. \angle \text{OAC} + y \tan. \angle \text{OBC} = \text{C}$, which is of the form $z = mx + ny + c$, where $m = \tan. \angle \text{CAX}$ and $n = \tan. \angle \text{CBY}$; and consequently $z = mx + c$ and $z = ny + c$ are the equations of the plane's intersections with XZ and YZ respectively.

Cor. 4. The angles which the plane makes with the axes X, Y, Z are the complements of the angles which the normal OD makes with the same axes, and consequently their sines can be computed, if we know either the plane's equation, or the distances from the origin at which it intersects the axes.

49. Conversely, an equation of the form $\frac{x}{\text{A}} + \frac{y}{\text{B}} + \frac{z}{\text{C}} = 1$ belongs to a plane surface; because, by assigning proper

values to the arbitrary constants, it may be made to coincide with the equation of any proposed plane.

50. There may be eight different planes, the equations of which shall contain the same constants; but the ambiguity is removed, as in Art. 39, by attending to their signs.

51. *Required the lines which are the intersections of a given plane with the co-ordinate planes.*

Let $\frac{x}{A} \pm \frac{y}{B} + \frac{z}{C} = 1$ be the plane's equation.

First, suppose $y = 0$, then it becomes $\frac{x}{A} + \frac{z}{C} = 1$, which is the equation of a right line in the plane XZ (7. 7.); and since it also belongs to the given plane, it is therefore the equation of their intersection.

Similarly it may be shown that $\frac{y}{B} + \frac{z}{C} = 1$, and $\frac{x}{A} + \frac{y}{B} = 1$ are the intersections of the plane with YZ and XY respectively.

This agrees with Art. 48. Cor. 3.

52. *Required the angles which a given plane makes with the co-ordinate planes.*

Let the plane's equation be $\frac{x}{A} + \frac{y}{B} + \frac{z}{C} = 1$; and let γ, β, α be the angles which it makes with the co-ordinate planes XY, XZ, YZ respectively; and let it cut the axes X, Y, Z in the points A, B, C (fig. 47.); then AB, AC, BC are its intersections with the planes XY, XZ, YZ.

Suppose a plane drawn through CO to intersect AB at right angles in N; then $\gamma = \angle$ CNO (Eu. 11. Def. 6.);

consequently $\cos.^2\gamma = \frac{ON^2}{CN^2} = \frac{OA^2 - AN^2}{CA^2 - AN^2} = \frac{A^2 - \frac{A^4}{A^2 + B^2}}{A^2 + C^2 - \frac{A^4}{A^2 + B^2}}$

$= \frac{A^2B^2}{A^2B^2 + A^2C^2 + B^2C^2} = \frac{1}{1 + \frac{C^2}{A^2} + \frac{C^2}{B^2}}$, or $\cos.\gamma = \ldots$

$\frac{1}{\sqrt{1 + \frac{C^2}{A^2} + \frac{C^2}{B^2}}}$.

Similarly $\cos.\beta = \dfrac{1}{\sqrt{1+\frac{\text{B}^2}{\text{A}^2}+\frac{\text{B}^2}{\text{C}^2}}}$; and $\cos.\alpha = \ldots$ $\dfrac{1}{\sqrt{1+\frac{\text{A}^2}{\text{B}^2}+\frac{\text{A}^2}{\text{C}^2}}}$.

If the plane's equation be under the form $\text{A}x + \text{B}y + \text{C}z = 1$; $\cos.\alpha = \dfrac{\text{A}}{\sqrt{\text{A}^2+\text{B}^2+\text{C}^2}}$, $\cos.\beta = \dfrac{\text{B}}{\sqrt{\text{A}^2+\text{B}^2+\text{C}^2}}$, and $\cos.\gamma = \dfrac{\text{C}}{\sqrt{\text{A}^2+\text{B}^2+\text{C}^2}}$; where $\text{A} = \dfrac{1}{\text{OA}}$, $\text{B} = \dfrac{1}{\text{OB}}$, & $\text{C} = \dfrac{1}{\text{OC}}$.

Cor. 1. $\cos.^2\alpha + \cos.^2\beta + \cos.^2\gamma = 1$.

Cor. 2. The angles which a plane makes with XY, XZ, YZ are the complements of the angles which it makes with the axes Z, Y, X respectively, and consequently the sines of the latter angles may be computed by the formulæ of this article.

53. *Required the angles which a plane makes with the co-ordinate planes in terms of the rectangular co-ordinates of the point where a perpendicular from the origin meets the plane.*

Making the same substitutions as in the preceding articles, the plane's equation is (47) $ax + by + cz = a^2 + b^2 + c^2$, or $\dfrac{ax}{a^2+b^2+c^2} + \dfrac{by}{a^2+b^2+c^2} + \dfrac{cz}{a^2+b^2+c^2} = 1$; whence, if the plane's equation be $\text{A}x + \text{B}y + \text{C}z = 1$, we have (Alg. 346.) $\text{A} = \dfrac{a}{a^2+b^2+c^2}$, $\text{B} = \dfrac{b}{a^2+b^2+c^2}$, ... $\text{C} = \dfrac{c}{a^2+b^2+c^2}$; and consequently $\cos.\alpha$, which . . . . . $= \dfrac{\text{A}}{\sqrt{\text{A}^2+\text{B}^2+\text{C}^2}}$, $= \dfrac{a}{\sqrt{a^2+b^2+c^2}}$; $\cos.\beta = \dfrac{b}{\sqrt{a^2+b^2+c^2}}$; and $\cos.\gamma = \dfrac{c}{\sqrt{a^2+b^2+c^2}}$.

Cor. 1. If OD is the perpendicular from the origin upon the plane, we have $\text{OD} = \sqrt{a^2+b^2+c^2}$ (40) $= \dfrac{1}{\sqrt{\text{A}^2+\text{B}^2+\text{C}^2}}$.

Cor. 2. The angles which a plane makes with YZ, XZ, XY

are equal to the angles which its normal makes with the axes X, Y, Z respectively.

Cor. 3. The plane's equation may be put under the form

$$\frac{ax}{\sqrt{a^2+b^2+c^2}}+\frac{by}{\sqrt{a^2+b^2+c^2}}+\frac{cz}{\sqrt{a^2+b^2+c^2}}=\sqrt{a^2+b^2+c^2},$$

or $x \cos.\alpha + y \cos.\beta + z \cos.\gamma = \text{OD}$.

54. *Def.* Through any proposed line in space let two planes be drawn; the one parallel to the axis Y, and consequently intersecting XZ at right angles (Eu. 11. 18.); and the other parallel to the axis X, intersecting YZ at right angles: these intersections are right lines (Eu. 11. 3.) situated in the planes XZ and YZ respectively; and their equations are called the *equations in space* of the proposed line.

The intersections are the orthographick projections of the line upon the co-ordinate planes XZ, YZ.

Cor. 1. The equations in space are of the form $\left.\begin{array}{l} x = az + \alpha \\ y = bz + \beta \end{array}\right\}$; where, if AC and EF are the projections upon the planes XZ, YZ (Vid. fig. 47.), $a = -\tan. \angle \text{ACO}$ and $\alpha = -\tan. \angle \text{ACO} \times \text{OC}$; also, $b = -\tan. \angle \text{OFE}$ and $\beta = -\tan. \angle \text{OFE} \times \text{OF}$.

Cor. 2. The proposed line has a third equation in space of the form $x = cy + \kappa$, which is its projection on the plane XY; but this is not an independent equation, as it may be obtained by eliminating z from the first two.

Cor. 3. The equations in space are also of the form $\left.\begin{array}{l} \frac{x}{a} + \frac{z}{c} = 1 \\ \frac{y}{b} + \frac{z}{c'} = 1 \end{array}\right\}$, where a and c, b and c' are the distances from the origin at which the projections on the planes XZ, YZ, cut the axes X and Z, Y and Z respectively.

Cor. 4. If a determinate value be assigned to either of the co-ordinates, the values of the remaining two can be obtained from the equations in space, and consequently the corresponding position of the point is determined.

55. The equations in space determine the position of the line; for it is evident, that if AC and EF be given, the position of a line which is the intersection of two planes drawn through AC, EF perpendicular to XZ, YZ is also given.

56. *The equations in space may also be deduced from the equations of the planes which are drawn through the proposed line perpendicular to* XZ, YZ.

For these equations are of the form $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, and $\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}=1$; of which the first is parallel to the axis Y, or $b=\infty$; and the second is parallel to the axis X, or $a=\infty$; consequently the equations become $\frac{x}{a}+\frac{z}{c}=1$ and $\frac{y}{b'}+\frac{z}{c'}=1$.

Cor. In the same manner the third equation may be shown to be of the form $\frac{x}{a}+\frac{y}{b}=1$.

57. *Def.* The position of a *curve* line in fixed space is also determined by its projections on two of the co-ordinate planes; and the equations of these projections are called its *equations in space.*

If the curve line be formed by the intersection of two given surfaces, as in the case of the curve of a groin, the equations of both the surfaces belong to the curve; and consequently, by eliminating one of the variables out of each equation, its equations in space will be obtained.

If three given surfaces intersect, there are three equations from which the position of their point of intersection may be determined.

Cor. The position of a point of the curve corresponding to any value of either of the co-ordinates is determined from the equations in space.

58. *Required the equations in space of a given circle situated in a given plane.*

Let α, β, γ be the co-ordinates of the centre of the circle; then the plane's equation is of the form

$$z' = mx' + ny' + c \text{ (48. Cor. 3.)}$$

therefore we have $\gamma = m\alpha + n\beta + c$

and consequently $z'-\gamma = m(x'-\alpha) + n(y'-\beta)$.

Let x, y, z be the co-ordinates of any point in the circumference of the circle, therefore $z-\gamma = m(x-\alpha)$ +

$n(y-\beta)$: also $(x-\alpha)^2+(y-\beta)^2+(z-\gamma^2)=(\text{rad.})^2=r^2$; wherefore, by substitution, $(x-\alpha)^2+(y-\beta)^2+\{m(x-\alpha)+n(y-\beta)\}^2=r^2$; which is the equation of an ellipse, the co-ordinates of whose centre are α, β.

59. *Def.* A curve of *double curvature* is one in which the generating point is continually changing, not only its direction, as in plane curves, but also the plane in which it is moving.

An instance of such curves is a *rhumb* line, which is a line drawn upon the surface of a globe intersecting all the meridians at a given angle.

This part of the subject will be considered in the second volume; but it may be observed here, that the definition in Art. 57 applies to these curves, and that their position and properties are made to depend upon the nature of the curves which are their orthographick projections on two of the co-ordinate planes.

60. *Def.* The inclination of two right lines in space is the angle made by two lines drawn from the origin parallel to them.

Cor. If the equations of the two lines are $\left.\begin{array}{l}x=az+\alpha\\ y=bz+\beta\end{array}\right\}$ and $\left.\begin{array}{l}x=mz+\mu\\ y=nz+\nu\end{array}\right\}$, their inclination is the angle made by two lines passing through O, whose equations are $\left.\begin{array}{l}x=az\\ y=bz\end{array}\right\}$ and $\left.\begin{array}{l}x=mz\\ y=nz\end{array}\right\}$.

61. *Required the angles which a given right line makes with the rectangular axes.*

Let its equations be $\left.\begin{array}{l}x=az+\alpha\\ y=bz+\beta\end{array}\right\}$; through O (fig. 47) draw OP $=r$ parallel to it; then OP makes the same angles with the axes.

Let v, v', v'' be the angles which the line makes with X, Y, Z respectively; then, since the equations in space of OP are $\left.\begin{array}{l}x=az\\ y=bz\end{array}\right\}$, therefore r^2, which $=x^2+y^2+z^2$, $=(a^2+b^2+1)z^2$; and $\cos.v''$, which $=\frac{z}{r}$, $=\frac{1}{\sqrt{a^2+b^2+1}}$.

Similarly, $\cos.v'=\frac{b}{\sqrt{a^2+b^2+1}}$ and $\cos.v=\frac{a}{\sqrt{a^2+b^2+1}}$.

Cor. 1. Since $\angle$ OPN, or its equal the $\angle$ PO*n*, is the complement of $\angle$ PON (fig. 38); therefore the angles which any line makes with the co-ordinate planes are the complements of the angles at which it is inclined to the axes at right angles to those planes; and consequently, if α, β, γ are the angles of its inclination to YZ, XZ, XY respectively, we have $\sin.\alpha = \dfrac{a}{\sqrt{a^2+b^2+1}}$, $\sin.\beta = \dfrac{b}{\sqrt{a^2+b^2+1}}$, and $\sin.\gamma = \dfrac{1}{\sqrt{a^2+b^2+1}}$.

Cor. 2. The sum of the squares of the sines of the angles which a right line makes with the co-ordinate planes is equal to the square of radius.

62. *Required the equations of condition in order that a right line shall be in a given plane.*

Let the required equations of the line be $\left.\begin{array}{l} x = az + \alpha \\ y = bz + \beta \end{array}\right\}$, and let x', y', z' be the co-ordinates of one of its points.

Also, let the plane's equation be under the form $Ax + By + Cz = 1$; then, since the line is in the plane, $Ax' + By' + Cz' = 1$; also, $\left.\begin{array}{l} x' = az' + \alpha \\ y' = bz' + \beta \end{array}\right\}$; wherefore, by substitution, $(Aa + Bb + C)\,z' + A\alpha + B\beta - 1 = 0$, whatever be the value of z'; whence (Alg. 347.) $Aa + Bb + C = 0$ and $A\alpha + B\beta - 1 = 0$ are the required equations of condition between a, α, b, β.

63. *Required the equations of condition that two planes may be parallel.*

The planes are parallel when their intersections with two of the co-ordinate planes are parallel.

Let their equations be $Ax + By + Cz = 1$, and $ax + by + cz = 1$; then, their intersections with XZ and YZ are (Art. 51) $\left.\begin{array}{l} Ax + Cz = 1 \\ ax + cz = 1 \end{array}\right\}$, and $\left.\begin{array}{l} By + Cz = 1 \\ by + cz = 1 \end{array}\right\}$, which lines are to be respectively parallel; consequently $\dfrac{A}{C} = \dfrac{a}{c}$ and $\dfrac{B}{C} = \dfrac{b}{c}$, or $a = \dfrac{Ac}{C}$ and $b = \dfrac{Bc}{C}$; whence, by substitution, $\dfrac{Ac}{C}x + \dfrac{Bc}{C}y + cz = 1$, or $\dfrac{c}{C}\left\{Ax + By + Cz\right\} = 1$, therefore

$$c = \frac{\text{C}}{\text{A}x + \text{B}y + \text{C}z}, \; b = \frac{\text{B}}{\text{A}x + \text{B}y + \text{C}z}, \; e = \frac{\text{A}}{\text{A}x + \text{B}y + \text{C}z},$$

which shows that the problem is *indeterminate*. If another condition be added, viz. that the required plane shall pass through a given point x', y', z', this will determine the values of a, b and c.

64. *If a normal to a plane which intersects one of the co-ordinate planes be projected upon that plane, its projection, produced if necessary, shall cut the line of intersection of the two planes at right angles.*

For, suppose a plane to be drawn through the normal at right angles to the co-ordinate plane; then this is the plane in which the normal is projected, and since it is at right angles to both the planes (Eu. 11. 18.), it is therefore (Eu. 11. 19.) at right angles to their line of intersection, or the line of intersection is at right angles to this plane, and consequently at right angles to the projected line which meets it in the same plane.

65. *Required the equations of condition that a line may be a normal to a given plane.*

Let the plane's equation be $\text{A}x + \text{B}y + \text{C}z = 1$; then its intersections with XZ, YZ are $\text{A}x + \text{C}z = 1$, and $\text{B}y + \text{C}z = 1$ (51), or $x = -\frac{\text{C}}{\text{A}}z + \frac{1}{\text{A}}$ and $y = -\frac{\text{C}}{\text{B}}z + \frac{1}{\text{B}}$.

Let the equations in space of the normal be $\left.\begin{matrix} x = az + \alpha \\ y = bz + \beta \end{matrix}\right\}$, which lines are perpendicular to the lines of intersection, and consequently (11) the requisite conditions are that $a = \frac{\text{A}}{\text{C}}$ and $b = \frac{\text{B}}{\text{C}}$.

If the normal is drawn from a given point (x', y', z') of the plane, its equations are $\left.\begin{matrix} x - x' = \frac{\text{A}}{\text{C}}(z - z') \\ y - y' = \frac{\text{B}}{\text{C}}(z - z') \end{matrix}\right\}$.

66. *Required the length of a normal drawn from a given point to a given plane.*

Let x', y', z' be the co-ordinates of the given point,
x, y, z the co-ordinates of that point where the normal meets the plane, then we have $\text{A}x + \text{B}y + \text{C}z = 1$ from the

plane's equation; and this, by adding and subtracting $\text{A}x' + \text{B}y' + \text{C}z'$, may be put under the form $\text{A}(x-x') + \text{B}(y-y') + \text{C}(z-z') + \text{K} = 0$, where $\text{K} = \text{A}x' + \text{B}y' + \text{C}z' - 1$.

In this equation substitute the values of $x - x'$, $y - y'$ deduced in the preceding article, and there results . . . . .

$$\frac{\text{A}^2 + \text{B}^2 + \text{C}^2}{\text{C}}(z-z') + \text{K} = 0; \text{ or } z - z' = \frac{-\text{CK}}{\text{A}^2 + \text{B}^2 + \text{C}^2}.$$

Similarly $y - y' = \dfrac{-\text{BK}}{\text{A}^2 + \text{B}^2 + \text{C}^2}$ and $x - x' = \dfrac{-\text{AK}}{\text{A}^2 + \text{B}^2 + \text{C}^2}$; therefore the length of the normal, which . . . .

$$= \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\ (41), = \frac{-\text{K}}{\sqrt{\text{A}^2 + \text{B}^2 + \text{C}^2}}$$

$$= \frac{1 - (\text{A}x' + \text{B}y' + \text{C}z')}{\sqrt{\text{A}^2 + \text{B}^2 + \text{C}^2}}.$$

67. *Required the equations of condition that two lines in space may be at right angles to each other.*

Let the equations in space of two lines drawn through the origin parallel to the proposed lines be $\left.\begin{array}{l} x = az \\ y = bz \end{array}\right\}$ and $\left.\begin{array}{l} x = mz \\ y = nz \end{array}\right\}$.

Let v, v', v'' and w, w', w'' be the angles which these lines make with X, Y, Z; then (61)

$$\cos. v = \frac{a}{\sqrt{a^2 + b^2 + 1}}, \quad \cos. v' = \frac{b}{\sqrt{a^2 + b^2 + 1}},$$

$$\cos. v'' = \frac{1}{\sqrt{a^2 + b^2 + 1}}; \text{ also,}$$

$$\cos. w = \frac{m}{\sqrt{m^2 + n^2 + 1}}, \quad \cos. w' = \frac{n}{\sqrt{m^2 + n^2 + 1}},$$

$$\cos. w'' = \frac{1}{\sqrt{m^2 + n^2 + 1}};$$

whence (43. Cor.) $\dfrac{am + bn + 1}{\sqrt{a^2 + b^2 + 1}\ \sqrt{m^2 + n^2 + 1}} = 0$; or $am + bn + 1 = 0$ is the required equation of condition.

68. *Required the angle which a given line makes with a given plane.*

From any point in the line draw a normal to the plane;

the angle made by the normal and the line is the complement of the required angle. (Eu. 11. Def. 5.)

Let $\left.\begin{array}{l} x = mz + \mu \\ y = nz + \nu \end{array}\right\}$ be the equations of the normal, and let the plane's equation be under the form $Ax + By + Cz = 1$; then (64.) $\left.\begin{array}{l} x = mz + \mu \\ y = nz + \nu \end{array}\right\}$ are at right angles to $Ax + Cz = 1$, and $By + Cz = 1$ respectively; consequently $m = \frac{A}{C}$ and $n = \frac{B}{C}$.

Let the equations of the line be $\left.\begin{array}{l} x = az + \alpha \\ y = bz + \beta \end{array}\right\}$.

Let v, v', v'' and w, w', w'' be the angles of inclination of the line, and of the normal to the axes X, Y, Z; then we have sine of the required angle

$$= \cos.v \cos.w + \cos.v' \cos.w' + \cos.v'' \cos.w'' \text{ (43.)}$$

$$= \frac{a}{\sqrt{a^2+b^2+1}} \times \frac{m}{\sqrt{m^2+n^2+1}} + \frac{b}{\sqrt{a^2+b^2+1}} \times \frac{n}{\sqrt{m^2+n^2+1}}$$

$$+ \frac{1}{\sqrt{a^2+b^2+1}} \times \frac{1}{\sqrt{m^2+n^2+1}}$$

$$= \frac{Aa + Bb + C}{\sqrt{a^2+b^2+1}\sqrt{A^2+B^2+C^2}}.$$

Cor. 1. The equation of condition that a right line may be parallel to a given plane is $Aa + Bb + C = 0$.

Cor. 2. If the given plane coincides with YZ, A which in this article $= \frac{1}{OA}$ (Vid. fig. 47.) $= \infty$; whence the sine of the line's inclination to YZ $= \frac{a}{\sqrt{a^2+b^2+1}}$, which agrees with Art. 61. Cor. 1.

69. *Required the angle which a line drawn from a given point in space to a given point in a given plane makes with the plane.*

Let x', y', z' and x'', y'', z'' be the co-ordinates of the given points; then, if $\left.\begin{array}{l} x = az + \alpha \\ y = bz + \beta \end{array}\right\}$ are the equations of the line, $a = \frac{x' - x''}{z' - z''}$ and $b = \frac{y' - y''}{z' - z''}$.

Also, let the plane's equation be $Ax + By + Cz = 1$; then it may be shown, as in the preceding article, that the sine of the required angle $= \dfrac{Aa + Bb + C}{\sqrt{a^2+b^2+1}\ \sqrt{A^2+B^2+C^2}}$; in which if we substitute their values for a and b, there will result the sine of the required angle in known terms.

70. *Required the angle of inclination of two given planes.*

Let $Ax + By + Cz = 1$ and $ax + by + cz = 1$ be the equations of the planes. From the origin draw two normal lines R and r; and their angle of inclination (U) is equal to the supplement of the required angle.

Let v, v', v'' and w, w', w'' be the angles of inclination of R and r to the axes X, Y, Z respectively; then

$$\left.\begin{array}{l} \cos. v = A.R,\ \cos. v' = B.R,\ \cos. v'' = C.R \\ \text{also, } \cos. w = a.r,\ \cos. w' = b.r,\ \cos. w'' = c.r \end{array}\right\}$$

whence (43.) $\cos. U = (Aa + Bb + Cc)\,Rr =$ (53. Cor. 1.) $\dfrac{Aa + Bb + Cc}{\sqrt{A^2+B^2+C^2}\ \sqrt{a^2+b^2+c^2}}$.

Cor. 1. The equation of condition that two planes shall be at right angles to each other is $Aa + Bb + Cc = 0$.

Cor. 2. Suppose the plane $ax + by + cz = 1$ to coincide with YZ; then $a = \infty$; consequently if V = the angle at which any plane, as $Ax + By + Cz = 1$, is inclined to YZ we have $\cos. V = \dfrac{A}{\sqrt{A^2 + B^2 + C^2}}$. Similarly, if V$'$ and V$''$ are the angles of its inclination to XZ, XY; $\cos. V' = \dfrac{B}{\sqrt{A^2+B^2+C^2}}$ and $\cos. V'' = \dfrac{C}{\sqrt{A^2+B^2+C^2}}$. Vid. Art. 52.

Cor. 3. If W, W$'$, W$''$ are the angles of inclination of any other plane, as $ax + by + cz = 1$, to the same co-ordinate planes, and U be the angle of inclination of the two planes, $\cos. U = \cos. V \cos. W + \cos. V' \cos. W' + \cos. V'' \cos. W''$.

For $\cos. U = A.R \times a.r + B.R \times b.r + C.R \times c.r$

$$= \frac{A}{\sqrt{A^2+B^2+C^2}} \times \frac{a}{\sqrt{a^2+b^2+c^2}} + \frac{B}{\sqrt{A^2+B^2+C^2}} \times \frac{b}{\sqrt{a^2+b^2+c^2}} + \frac{C}{A\sqrt{{}^2+B^2+C^2}} \times \frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$= \cos. V \cos. W + \cos. V' \cos. W' + \cos. V'' \cos. W''.$$

71. *Required to draw a line through two given points in space.*

Let the co-ordinates of the two given points be x', y', z' and x'', y'', z''; then the equations of the required line are of the form $\left.\begin{array}{l} x = az + \alpha \\ y = bz + \beta \end{array}\right\}$; consequently we have $\left.\begin{array}{l} x' = az' + \alpha \\ y' = bz' + \beta \end{array}\right\}$ and $\left.\begin{array}{l} x'' = az'' + \alpha \\ y'' = bz'' + \beta \end{array}\right\}$; whence, by elimination, ...
$\left.\begin{array}{l} x - x' = a(z - z') \\ y - y' = b(z - z') \end{array}\right\}$ and $\left.\begin{array}{l} x' - x'' = a(z' - z'') \\ y' - y'' = b(z' - z'') \end{array}\right\}$; wherefore $a = \dfrac{x' - x''}{z' - z''}$ and $b = \dfrac{y' - y''}{z' - z''}$, and by substitution, ...

$$\left.\begin{array}{l} x - x' = \dfrac{x' - x''}{z' - z''}(z - z') \\[2ex] y - y' = \dfrac{y' - y''}{z' - z''}(z - z') \end{array}\right\}$$ which are the equations of the required line.

72. *Required to draw a line parallel to a given line.*

Let $\left.\begin{array}{l} x = az + \alpha \\ y = bz + \beta \end{array}\right\}$ be the equations of the given line;
$\left.\begin{array}{l} x = mz + \mu \\ y = nz + \nu \end{array}\right\}$ the required equations.

The lines are parallel when their projections on XZ and YZ are parallel, *i. e.* when $m = a$ and $n = b$ (60. Cor.); hence the required equations become $\left.\begin{array}{l} x = az + \mu \\ x = bz + \nu \end{array}\right\}$ where μ and ν are indeterminate.

If it be also required to draw the line through a given point whose co-ordinates are x', y', z'; we have $\left.\begin{array}{l} x' = az' + \mu \\ y' = bz' + \nu \end{array}\right\}$;
and, by elimination, $\left.\begin{array}{l} x - x' = a(z - z') \\ y - y' = b(z - z') \end{array}\right\}$ are the equations of the required line, which cannot be made to satisfy any new condition.

73. *Required to draw a right line through a given point making given angles with the rectangular axes.*

Let the given point be (x', y', z') and v, v', v'' the angles which the required line makes with X, Y, Z respectively.

If the equations of the required line are $\left.\begin{array}{l} x = az + \alpha \\ y = bz + \beta \end{array}\right\}$,

we have (61.) $\cos. v = \dfrac{a}{\sqrt{a^2+b^2+1}}$, $\cos. v' = \dfrac{b}{\sqrt{a^2+b^2+1}}$

and $\cos. v'' = \dfrac{1}{\sqrt{a^2+b^2+1}}$; whence $a = \dfrac{\cos. v}{\cos. v''}$, $b = \dfrac{\cos. v'}{\cos. v''}$:

and since $\left.\begin{array}{l} x' = az' + \alpha \\ y' = bz' + \beta \end{array}\right\}$, the required equations are . .

$$\left.\begin{array}{l} x - x' = \dfrac{\cos. v}{\cos. v''}(z - z') \\ y - y' = \dfrac{\cos. v'}{\cos. v''}(z - z') \end{array}\right\}.$$

Cor. The equations of a right line drawn through (x', y', z') and making the angles α, β, γ with the planes YZ, XZ, XY

respectively are $\left.\begin{array}{l} x - x' = \dfrac{\sin. \alpha}{\sin. \gamma}(z - z') \\ y - y' = \dfrac{\sin. \beta}{\sin. \gamma}(z - z') \end{array}\right\}$ (60. Cor. 1).

74. *Required to draw a plane through three given points.*

Let a, b, c; a', b', c'; and a'', b'', c'' be the three points; assume $z = mx + ny + d$ for the plane's equation (48. Cor. 3.) which will be determined, if m, n and d are found.

Now we have $\left.\begin{array}{l} z = mx + ny + d \\ c = ma + nb + d \\ c' = ma' + nb' + d \end{array}\right\}$ . . . .

whence $z - c = m(x - a) + n(y - b)$ (α) . .

and $c' - c = m(a' - a) + n(b' - b)$ (β).

Also $c'' = ma'' + nb'' + d$, therefore $c'' - c = m(a'' - a) + n(b'' - b)$ which combined with (β) gives . . . .

$m = \dfrac{(c' - c)(b'' - b) - (c'' - c)(b' - b)}{(a' - a)(b'' - b) - (a'' - a)(b' - b)}$ and

$n = \dfrac{(c' - c)(a'' - a') - (c'' - c)(a' - a)}{(b' - b)(a'' - a) - (b'' - b)(a' - a)}$; which values of m and n substituted in (α) will give the required equation.

Hence it appears that any number of points less than three is insufficient for determining a plane's position; and any number greater than three superfluous.

75. *Required to draw a plane through a given line and a given point.*

Let $\left.\begin{array}{l} x = az + \alpha \\ y = bz + \beta \end{array}\right\}$ be the given equations of the line;

(x', y', z') the given point; and $\dfrac{x}{\text{A}} + \dfrac{y}{\text{B}} + \dfrac{z}{\text{C}} = 1$ the required plane.

Draw a normal to the plane, then its equations are . . .

$\left.\begin{array}{l} x = \dfrac{\text{C}}{\text{A}} z + \mu \\ y = \dfrac{\text{C}}{\text{B}} z + \nu \end{array}\right\}$ (65); and this line is at right angles to the given line (Eu. 11. Def. 3.), consequently $\dfrac{\text{A}}{\text{C}} = -a$, $\dfrac{\text{B}}{\text{C}} = -b$; and by substitution, the plane's equation becomes $z = \dfrac{x}{a} + \dfrac{y}{b} + \text{C}$; and since it passes through (x', y', z'), the required plane is $z - z' = \dfrac{1}{a}(x - x') + \dfrac{1}{b}(y - y')$.

76. *Required to draw a plane through a given point intersecting a given right line at right angles.*

Let $\left.\begin{array}{l} x = az + \alpha \\ y = bz + \beta \end{array}\right\}$ be the equations of the given line; x', y', z' the given point and $\text{A}x + \text{B}y + \text{C}z = 1$, the required equation of the plane.

Since the plane passes through x', y', z' we have $\text{A}(x-x') + \text{B}(y-y') + \text{C}(z-z') = 0$ (a). Also, the intersections of the plane with XZ and YZ are $\left.\begin{array}{l} \text{A}x + \text{C}z = 1 \\ \text{B}y + \text{C}z = 1 \end{array}\right\}$ (51), and these are at right angles to the lines $\left.\begin{array}{l} x = az + \alpha \\ y = bz + \beta \end{array}\right\}$ respectively (64.); wherefore $a = \dfrac{\text{A}}{\text{C}}$ and $b = \dfrac{\text{B}}{\text{C}}$, and by substitution in (a) and dividing by C, there results $a(x-x') + b(y-y') + z - z' = 0$ for the required equation.

If the point (x', y', z') is in the given line, we have

$\left.\begin{array}{l} x' = az' + \alpha \\ y' = bz' + \beta \end{array}\right\}$ and the equation of the required plane becomes $ax + by + z - (a^2 + b^2 + 1)z' - (a\alpha + b\beta) = 0$.

77. ***Required to draw a plane through a given point making given angles with the co-ordinate planes.***

Let x', y', z' be the co-ordinates of the given point,

γ, β, α the angles which the plane makes with XY, XZ, YZ.

Then (50. Cor. 1) $x \cos.\alpha + y \cos.\beta + z \cos.\gamma = \text{D}$ where D is indeterminate; but since the plane passes through x', y', z'; we have $x' \cos.\alpha + y' \cos.\beta + z' \cos.\gamma = \text{D}$; and by elimination $(x - x') \cos.\alpha + (y - y') \cos.\beta + (z' - z) \cos.\gamma = 0$ is the required equation.

CHAPTER VIII.

Tangents, normals, asymptotes, and tangent planes.

1. Definitions.

(1.) A *tangent* is a right line which meets a curve, but does not cut it in the neighbourhood of the touching point.

The touching point is here supposed not to be a *singular* point. Vid. Ch. 12.

(2.) The *subtangent* is that part of the axis of the abscissæ which is intercepted between the ordinate and the tangent.

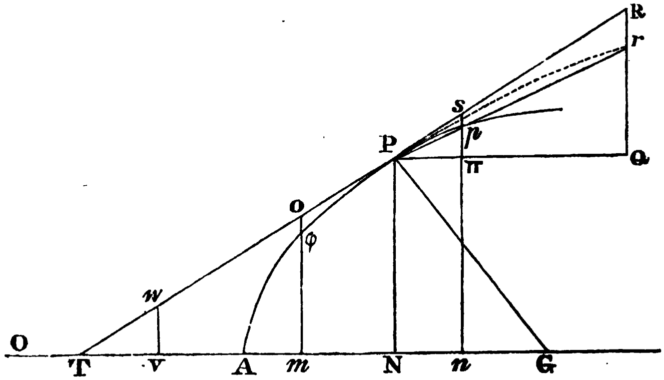
(3.) A *normal* is a perpendicular to the curve, intercepted by the axis.

(4.) The *subnormal* is that part of the axis which is intercepted between the ordinate and the normal.

Thus in the figure, PT is the tangent; NT the subtangent; PG the normal; NG the subnormal.

(5.) *Asymptotes* are right lines or curves, which, cutting one of the axes at a finite distance from the origin, by increasing the abscissa indefinitely, may be made to approach nearer to the curve than by any assignable distance.

2. *Required to find geometrical representations for the fluxions of a curve and of its co-ordinates.*



Let AN, NP be the co-ordinates; AP the intercepted arc.

Draw PQ parallel to the axis, and take it of a finite and determinate magnitude to represent the fluxion of AN; let TPR be a tangent to the curve at P, meeting the axis of the abscissæ in T; draw QR parallel to the axis of the ordinates meeting the tangent in R; then the sides of the triangle PQR shall represent the required fluxions.

For draw the ordinate $p\pi n$ near to PN; produce it to meet the tangent in s; join Pp, and produce it to meet QR in r: then as pn moves towards PN, there may always be drawn a curve Pr similar to Pp, which has the same tangent PR (7. 29.).

Now inc. AN : inc. PN : inc. AP : : Pπ : $p\pi$: curve Pp,
: : PQ : Qr : curve Pr;

and to obtain the limit of these ratios, suppose pn to move towards PN; then ultimately the $\angle$ RPr vanishes (7. 27.), and therefore instead of Qr and curve Pr, we may substitute in the limit QR and PR; whence by the definition, Ch. 1. Art. 7. d.AN : d.PN : d.AP : : PQ : QR : PR.

PQR is called the fluxional triangle of the curve at P.

Cor. 1. Since Rr ultimately vanishes, the limit of $s\pi : p\pi$ is a ratio of equality, and consequently the first fluxions of the ordinate of the curve and of the tangent are equal.

Cor. 2. Let ON $= x$, NP $= y$, AP $= s$ | then, when the co-ordinates are rectangular, PR^2 = PQ^2 + QR^2, or $ds^2 = dx^2 + dy^2$, or . . . .

$$\frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}} = \text{by substitution } \sqrt{1+p^2}.$$

Cor. If the co-ordinates are not rectangular but inclined at an angle a, $ds^2 = dx^2 + dy^2 - 2\cos.a dxdy$ (Trig. p. 24), or

$$\frac{ds}{dx} = \left\{\frac{dy^2}{dx^2} - 2\cos.a\frac{dy}{dx} + 1\right\}^{\frac{1}{2}} = (p^2 - 2\cos.ap + 1)^{\frac{1}{2}}.$$

3. *Given a curve's equation; required the equation of the tangent at any point of the curve.*

Let x', y' be the co-ordinates of the tangent,
x, y those of the curve at P.

Since the equation of the tangent is of the form $y' = ax' + b$ (7, 7), and that it passes through the point (x, y); therefore we have, by elimination, $y' - y = a(x' - x)$; and $a = \tan. \angle$ PTN $= \tan. \angle$ RPQ $= \frac{dy}{dx}$; wherefore the re-

quired equation is $y' - y = \frac{dy}{dx}(x' - x') = p(x' - x)$.

Hence if the equation of the curve is of the form $y = \mathrm{F}x$, by differentiation, $\frac{dy}{dx}$ and consequently the equation of the tangent may be obtained.

The equation may also be found if y is an implicit function of x.

4. *Required the equation of the normal.*

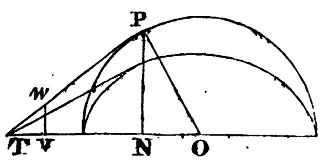
Let x', y' be the co-ordinates of the normal PG; then its equation is of the form $y' = ax' + \alpha$, where $a = -\tan. \angle \text{PGN}$ $= (\text{A}) - \tan. \angle \text{PRQ} = -\frac{dx}{dy}$; and it passes through (x, y), therefore the required equation is $y' - y = -\frac{dx}{dy}(x' - x)$ $= -\frac{1}{p}(x' - x)$.

If then the curve's equation be given, and we can resolve it, we can find the equations of the tangent and of the normal of the point P in terms of either of its co-ordinates.

If the angle at which the axes are inclined be changed, the equations of the tangent and normal remain unaltered.

Examples.

Ex. 1. To draw a tangent and a normal to any point of a given circle.



$x^2 + y^2 = r^2 \therefore p = -\frac{x}{y}$, which substituted in the formula Art. 2, gives $y' - y$ $= -\frac{x}{y}(x' - x)$ or $xx' + yy'$ $= x^2 + y^2 = r^2$, or $y' = -\frac{x}{y}x' + \frac{r^2}{y}$, which is the equation of the required line TP.

It gives the following property, ON × OV + NP × VW = OP², a constant quantity.

To find the normal; since $\frac{-dx}{dy} = \frac{y}{x}$, its equation is

$y' - y = \frac{y}{x}(x' - x)$, or $y' = \frac{yx'}{x}$, which shows that the normal passes through O.

The position of P or the value of its co-ordinates x and y is supposed to be given.

Ex. 2. To draw a tangent and a normal to an ellipse.

$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \therefore p = -\frac{b^2}{a^2} \cdot \frac{x}{y}$, and the required equation is $y' - y = -\frac{b^2}{a^2} \cdot \frac{x}{y}(x' - x)$, or $y' = -\frac{b^2 x}{a^2 y}x' + \frac{b^2}{y}$.

To find the normal; $y' - y = \frac{a^2 y}{b^2 x}(x' - x)$, or $y' = \frac{a^2 y}{b^2 x}x' - \frac{a^2 - b^2}{b^2}y$.

Ex. 3. An hyperbola between its asymptotes.

$$xy = ab \therefore y = abx^{-1} \therefore p = -\frac{ab}{x^2} = -\frac{y}{x},$$

$$\therefore y' - y = -\frac{y}{x}(x' - x), \text{ or } y' = -\frac{y}{x}x' + 2y.$$

5. *Required the subtangent and the subnormal of a curve.*

The subtangent is NT; and from Δ', $\text{TN} = \frac{\text{PN} \times \text{PQ}}{\text{QR}} = \frac{ydx}{dy} = \frac{y}{p}$.

The subnormal or $\text{NG} = \frac{\text{PN} \times \text{RQ}}{\text{PQ}} = \frac{ydy}{dx} = yp$.

These may be found in terms either of x or of y from the curve's equation.

Cor. 1. Hence arises a method of constructing either for the tangent or for the normal.

For having calculated NT or NG by the formulæ of this article, join TP or GP, which will be the required lines.

Cor. 2. $\text{PT} = \frac{yds}{dy}$.

Also draw AY perpendicular to PT, and $\text{AY} = \frac{ydx - xdy}{ds}$.

Cor. 3. If two curves have a common abscissa, and their

corresponding ordinates be always in a given ratio; they will have the same subtangent.

For $\frac{ny.dx}{ndy} = \frac{ydx}{dy}$.

6. *The values of the subtangent and of the subnormal may also be deduced from the equations of Arts. 3 and 4.*

For in these equations make $y' = 0$, therefore

$-y = p(x' - x)$, or $x - x' = \frac{y}{p}$, which therefore $=$ TN.

Also in the normal equation, $-y = -\frac{1}{p}(x' - x)$, or $x' - x = yp =$ NG.

7. *Examples.*

In these examples we suppose the origin of the co-ordinates to coincide either with the vertex or with the centre of the curve.

Ex. 1. Required the subtangent and the subnormal of the *Apollonian parabola.*

$y^2 = ax \therefore 2ly = la + lx \therefore \frac{2dy}{y} = \frac{dx}{x}$ and $\frac{y}{p} = 2x$.

To construct for the tangent; take NT $=$ 2AN, and join TP, which will be the tangent.

Again, $yp = \frac{a}{2}$, a constant quantity; take then NG $= \frac{1}{2}$ the latus rectum, and PG will be perpendicular to the curve at P.

Ex. 2. A *circle.*

$y^2 = 2ax - x^2 \therefore 2ly = l.(2ax - x^2) \therefore \frac{p}{y} = \frac{a-x}{2ax-x^2}$

$\therefore \text{NT} = \frac{2ax-x^2}{a-x}$.

Also, $yp = a - x$, or G coincides with the centre of the circle.

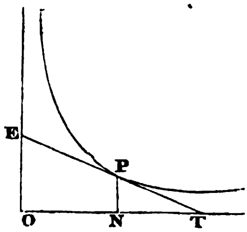
Ex. 3. An *ellipse.*

$y^2 = \frac{b^2}{a^2}(2ax - x^2) \therefore 2ly = l.\frac{b^2}{a^2} + l.(2ax - x^2)$; whence

$\text{NT} = \dfrac{2ax - x^2}{a - x}$, the same result as for the circle described upon its axis major; which shows that the circular and elliptic arcs having the same abscissa will have the same subtangent.

Also, $yp = \dfrac{b^2}{a^2} \cdot (a - x)$. This result proves that NC : NO : : $\text{BO}^2 : \text{AO}^2$.

Ex. 4. A rectangular hyperbola between its asymptotes.



$xy = a^2 \therefore y = a^2 x^{-1} \therefore p = -\dfrac{a^2}{x^2}$

$\therefore \text{NT} = \dfrac{y}{p} = -\dfrac{x^2 y}{a^2} = -x$; the negative sign shows that NT and NO are to be measured in opposite directions from N. Take NT = ON, join TP which will touch the hyperbola at P.

Cor. Produce TP to meet the asymptote in E, then $\triangle$ TEO is constant and $= 2a^2$.

Ex. 5. To the versed sine of a circular arc there is erected an ordinate which is a fourth proportional to the versed sine, the sine, and the diameter: the traced out curve is called the "witch;" and it is required to calculate the subtangent.

Here $y = \dfrac{a\sqrt{ax - x^2}}{x} = a\sqrt{ax^{-1} - 1} \therefore p = \dfrac{-\frac{1}{2}a^2}{x^2\sqrt{ax^{-1} - 1}}$

$= \dfrac{-\frac{1}{2}a^2}{x\sqrt{ax - x^2}} \therefore \text{NT} = \dfrac{y}{p} = -\dfrac{2}{a}(ax - x^2).$

Ex. 6. Let the curve be $lx = \dfrac{a}{y}$, or $x = e^{\frac{a}{y}}$.

$\dfrac{dx}{x} = \dfrac{-ady}{y^2} \therefore \text{NT} = \dfrac{y}{p} = -a\dfrac{x}{y} = -xlx.$

Ex. 7. Let the curve be $ax^2 + xy^2 + x^3 - y^3 = 0$.

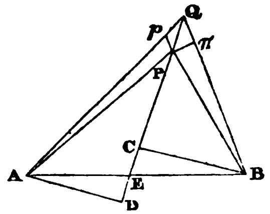
Here y is an implicit function of x; and by differentiating, we shall have the subtangent $= \dfrac{3y^3 - 2xy^2}{y^2 + 3x^2 + 2ax}$, which cannot be found in terms of one of the co-ordinates alone, unless we can solve the original equation.

Ex. 8. Let the equation be $ax^3 + x^3y - ay^3 = 0$.
Differentiating $(3ax^2 + 3x^2y)dx + (x^3 - 3ay^2)dy = 0$ $\therefore$

$$\frac{y}{p} = \frac{3ay^3 - x^3y}{3ax^2+3x^2y} = \frac{3ax^3+2x^3y}{3ax^2+3x^2y} = \frac{x.(3a+2y)}{3(a+y)} \dots\dots$$

$$= \frac{a^{\frac{1}{3}}}{3} \times \frac{3ay+2y^2}{(a+y)^{\frac{4}{3}}}.$$

Ex. 9. To draw a tangent to a curve traced by the vertex of a triangle whose base is constant, and the angles at the base in a constant ratio.



Let the constant ratio be that of $n : 1$; take PQ a small portion of the arc, which in the limit coincides with the tangent at P; draw BC, AD perpendicular to the tangent at P and $\text{P}p$, Pπ perpendicular to AQ, BQ.

Then $n : 1$ ultimately $= \frac{\text{P}p}{\text{AP}} : \frac{\text{P}\pi}{\text{BP}} = \frac{\sin. \angle \text{PQ}p}{\text{AP}} : \frac{\sin. \angle \text{PQ}\pi}{\text{BP}}$

$= \frac{\text{AD}}{\text{AP}^2} : \frac{\text{BC}}{\text{BP}^2}$ $\therefore n \times \frac{\text{BC}}{\text{BP}^2} = \frac{\text{AD}}{\text{AP}^2}$ $\therefore n \times \frac{\text{AP}^2}{\text{BP}^2} = \frac{\text{AD}}{\text{BC}} = \frac{\text{AE}}{\text{BE}}$, which determines the point E in which the tangent intersects the base.

8. PRAXIS.

Ex. 1. Find the subtangent and subnormal of a parabola whose equation is $a^{n-1}x = y^n$, $\text{NT} = nx$ and $\text{NG} = \frac{y^2}{nx}$.

2. The subtangent of the Apollonian hyperbola . . . . . $= \frac{2ax+x^2}{a+x}$.

3. The subtangent of an hyperbola between the asymptotes whose equation is $y^{n-1}x = a^n$.

4. The general equations to an ellipse and an hyperbola of any degree are $\frac{2ay^{m+n}}{b} = x^m(2a \mp x)^n$ and . . . .

$$\text{NT} = \frac{(m+n)(2ax \mp x^2)}{2ma \mp (m+n)x}.$$

9. *Conversely, if the subtangent or subnormal be given, the curve's equation can be found.*

Ex. The curve whose subnormal is a constant quantity is a parabola.

For $\frac{ydy}{dx} = \frac{a}{2}$ $\therefore ydy = \frac{adx}{2}$, and integrating $y^2 = ax$. *

10. *Required the angle which a tangent at any point of the curve makes with the axis.*

Tan. $\angle$ PTN = tan. $\angle$ QPR $= \frac{dy}{dx} = p$, which may be found in terms of x and y from the equation of the curve.

Cor. 1. Hence the tangent of the angle which the curve makes with the axis at the origin equals (p) when $x = 0$.

Cor. 2. If the origin is not at the point of intersection, we must find (p) when $y = 0$.

Ex. 1. In the Apollonian parabola, required the angle which the tangent at the extremity of the latus rectum makes with the axis.

$y^2 = ax$ $\therefore \frac{dy}{dx} = \frac{a}{2y}$ $\therefore (p) = 1$, or required $\angle = 45°$.

Ex. 2. Find the subtangent of a curve whose equation is $y = \frac{x}{1+x^2}$; and also the angle at which it cuts the axis.

$ly = lx - l(1+x^2)$ $\therefore \frac{dy}{y} = \frac{dx}{x} - \frac{2xdx}{1+x^2} = \frac{(1-x^2)dx}{x.(1+x^2)}$ $\therefore$

NT $= \frac{y}{p} = \frac{x.(1+x^2)}{1-x^2}$.

Also, when $x = 0$, $y = x = 0$, whence tan. required

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* Mathematicians first directed their attention to maxima and minima problems; but more especially to the method of drawing tangents; and as the inverse of this latter class of problems always require the integration of fluxional equations, we see the reason of their giving the name of the *inverse method of tangents* to what is now called the inverse method of fluxions, or the integral calculus.

$\angle = (p) = \frac{y.(1-x^2)}{x.(1+x^2)} = \frac{1-x^2}{(1+x^2)^2} = 1 \therefore$ required $\angle = 45^\circ$.

Ex. 3. Required the angle which a circular arc makes with its chord.

Let the chord $= c$
$b =$ its distance from centre, then it may be shown that the equation is $y^2 + 2by = cx - x^2$

$\therefore 2(y+b)dy = (c-2x)dx \therefore p = \frac{c-2x}{2.(y+b)} \therefore$ tan. required

$\angle = (p) = \frac{\frac{1}{2}c}{b}$.

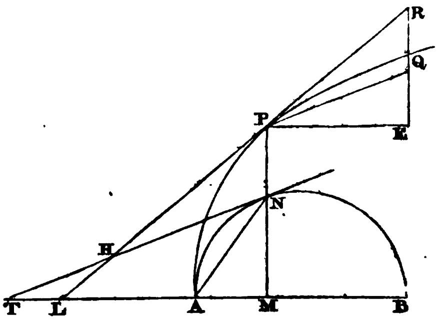
Ex. 4. $ax^3 + x^3y - ay^3 = 0$; required the angle at which the curve cuts the axis.

Since $x^3 = \frac{ay^3}{a+y} \therefore$ when $x = 0$ and $y = 0$, x^3 in the limit, $= \frac{ay^3}{a} = y^3 \therefore x = y \therefore (p) = 1$, or the $\angle = 45^\circ$.

The value of p in this example is $\frac{3ax^2 + 3x^2y}{3ay^2 - x^3}$, which when $x = 0$, $y = 0$, takes the indeterminate form of $\frac{0}{0}$; hence, proceeding as in Ch. 5. 21, we have . . . .
$(p) = \frac{6ax + 6xy + 3x^2(p)}{6ay(p) - 3x^2} = \frac{6a + 6y + 12x(p)}{6a(p)^2 - 6x}$, $\therefore$ the equation for determining (p) is $a\ (p)^3 - 3x(p) = a$, or $(p) = 1$.

When (p) takes the form of $\frac{0}{0}$, the curve may have more than one tangent; in which case there must be more than one branch at the proposed point. The student may recur to the examples of Ch. 5. Art. 21. In Ex. 1, the curve has two branches at the origin, which are equally inclined to the axis x: in Ex. 3, there are two branches, one of which touches the axis, and the other cuts it at right angles; and in the same manner the position of the branches of the curve at the origin may be determined in all the remaining examples.

11. *Required to draw the tangent at a given point of a given curve which has a curvilinear abscissa.*



Let AN be the curvilinear abscissa, NP the ordinate; draw the tangent PH meeting the tangent NT in H; then NH is P's subtangent. Construct the fluxional triangle PQR as in Art. 2, except that PQ is now to be drawn parallel to TN; then it may be shown as in that article, that PQ $= d$. AN, QR $= d$. NP and PR $= d$. AP.

AM $= x$ MN $= y$ AN $= v$ NP $= w$ MP $= z$	From Δ's HNP, PQR; NH $= \frac{wdv}{dw}$, which can be found from the curve's equation; for v is a function of x and y (2. Cor. 2.)

Cor. Produce PH to meet the axis of the curve AMB in L; then to find ML, draw PE perpendicular to RQE and (Δ's), we have ML $=$ PM $\times \frac{\text{PE}}{\text{RE}} = \frac{zdx}{dz}$.

Ex. 1. Let ANB be a circle, and NP $=$ AN.
$v = w$ $\therefore$ NH $= w =$ NP $=$ NA, or the locus of H is the involute of the circle.

To find ML, we have PM $=$ AN $+$ NM $\therefore dz = \frac{adx}{\sqrt{2ax - x^2}}$

$+\frac{(a-x)dx}{\sqrt{2ax-x^2}} = \frac{\sqrt{2a-x}}{\sqrt{x}}dx$ $\therefore$ ML $= \frac{zdx}{dz} = \frac{z\sqrt{x}}{\sqrt{2a-x}}$ . . . .

$= \frac{zx}{\sqrt{2ax-x^2}}$ $\therefore$ ML : MP :: AM : MN, or the tangent at P is parallel to the chord AN.

Ex. 2. Let AN be a parabola, AB its latus rectum, and suppose NP^2 $=$ AM $\times$ MB.
$y^2 = 2bx$ the equation of the parabola, then (Ch. 9. 4. Ex. 3.)

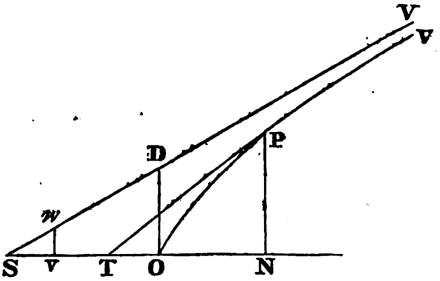
$$dv = \frac{(y^2+b^2)^{\frac{1}{2}}dy}{b} = \frac{(2bx+b^2)^{\frac{1}{2}}}{b} \times \frac{b^{\frac{1}{2}}dx}{\sqrt{2x}} = \left(\frac{2x}{2x}\frac{b}{}\right)^{\frac{1}{2}}dx. \text{ Also}$$

$w = \sqrt{x(2b-x)}$ $\therefore$ $lw = \frac{1}{2}l(2bx - x^2)$ $\therefore$ $\frac{dw}{w} = \frac{(b-x)dx}{2bx - x^2}$

$\therefore$ NH $= \frac{wdv}{dw} = \left(\frac{2x+b}{2x}\right)^{\frac{1}{2}} \times \frac{2bx - x^2}{b-x} = \frac{2b-x}{b-x} \cdot \left(x^2 + \frac{bx}{2}\right)^{\frac{1}{2}}$.

Ex. 3. Let AN be a parabola whose latus rectum $= p$, and take NP : AN : : e : 1, then ML $= \frac{2esx^{\frac{1}{2}} + 2p^{\frac{1}{2}}x}{e\sqrt{p+4x} + p^{\frac{1}{2}}}$, where $s =$ AN.

12. *Required to find the equation of the rectilinear asymptote of a given curve.*



An asymptote may be considered as the limit of the tangent, when the abscissa is indefinitely increased.

Let PT be a tangent, sv the required asymptote; its equation is of the form $y' =$ Ax' + B. Suppose that by means of the binomial or any other theorem, that the given equation of the curve can be put under the form $y = ax + b + \frac{c}{x} + \frac{e}{x^2} +$ &c.; this, when x is infinite, becomes $y = ax + b$, which is the equation of the tangent at an infinite distance; and consequently the required equation of the asymptote is $y' = ax' + b$.

Otherwise. A $=$ tan. $\angle$ s (7. 7) $=$ tan. $\angle$ T when x is infinite $= (p)$; but since (ex. hyp.) $y = ax + b + \frac{c}{x} +$ &c., $\therefore$ differentiating, $p = a - \frac{c}{x^2} -$ &c.; therefore $(p) = a =$ A.

Next to find B; B $=$ SO $\times$ tan. $\angle$ s (7. 7) $= a \times$ TO when x is infinite $= a \left\{ \text{TN} - \text{ON} \right\} = a \left\{ \frac{y}{p} - x \right\} =$ $y - ax = b + \frac{c}{x} +$ &c. $= b$ when x is infinite.

Cor. 1. The equation of the asymptote may be put under the form of $y' = (p)x' + (y - px)$ where the brackets denote the value of the functions when $x = \infty$.

Cor. 2. Since the equation of the tangent is $y' - y = p(x' - x)$ (Art. 3), or $y' = px' + y - px$, it appears that the asymptote is the limit of the tangent.

Cor. 3. Hence to determine whether a curve has a rectilinear asymptote, calculate the value of $\frac{y-px}{p}$ or $\frac{y}{p} - x$ when $x = \infty$, and if this be finite, there is an asymptote whose inclination to the axis $= \tan.^{-1}(p)$.

Cor. 4. If in the developement of y, we do not neglect $\frac{c}{x}$, there arises an equation to a conic section, which may be considered as a curvilinear asymptote tending to touch the curve more intimately than the line $y' = ax' + b$.

By including more terms of the series, we may have a succession of curvilinear asymptotes, each tending to touch the curve more intimately than those which precede it.

13. *Examples.*

Ex. 1. The Apollonian hyperbola.

The equation is $y = \frac{b}{a}\sqrt{x^2 - a^2} = \frac{bx}{a}\left(1 - \frac{a^2}{x^2}\right)^{\frac{1}{2}} \ldots$ $= \frac{bx}{a}\left\{1 - \frac{a^2}{2x^2} - \frac{a^4}{8x^4} - \text{\&c.}\right\} = \frac{bx}{a} - \frac{ab}{2x} - \text{\&c.}$ $\therefore$ the equation of the asymptote is $y' = \frac{b}{a}x'$; or the asymptote passes through the centre inclined to the axis at an angle whose tangent $= \frac{b}{a}$.

Otherwise. $p = \frac{bx}{a\sqrt{x^2-a^2}} \therefore (p) = \frac{b}{a}$; also $(y) = \frac{bx}{a} = (p)x \therefore (y - px) = 0$, or the equation is $y' = \frac{bx'}{a}$.

If the origin is at the vertex of the hyperbola, the equation of the asymptote is $\frac{y'}{b} - \frac{x'}{a} = 1$.

If the axis minor of the hyperbola is equal to its axis major, it is said to be *equilateral;* in this case tan. $\angle$ S $= 1$ and $\angle$ S $= 45°$. It appears also from the equation of the curve, that its four branches are symmetrical on the axes X and Y; consequently the asymptotes of an equilateral hyperbola intersect at right angles.

Ex. 2. $ay^2 + (b + cx)y + ex^2 + fx + g = 0$; the canonical equation of the second degree.

Developing y either by Maclaurin's theorem or by the rule for extracting the square root, it will appear that there is an asymptote only when c^2 is greater than $4ae$, *i.e.* when the curve is an hyperbola.

Ex. 3. $y^3 = ax^2 + x^3$.

When $x = \infty, y = \infty$ $\therefore$ from the equation $(y) = x$.

Also, $p = \frac{2ax + 3x^2}{3y^2} \therefore (p) = 1$; and $y - px = y - \frac{2ax^2 + 3x^3}{3y^2}$

$= \frac{ax^2}{3y^2} \therefore (y - px) = \frac{a}{3}$, or the required equation is . . . .

$y' = x' + \frac{a}{3}$.

Hence to draw the asymptote take OS $= \frac{a}{3}$, and through S draw SV making $\angle$ OSV $= 45°$.

Ex. 4. $\frac{a^2 y^{m+n}}{b^2} = x^m(2a + x)^n$.

$l\frac{a^2}{b^2} + (m + n)ly = mlx + nl(2a + x) \therefore \frac{p}{y} =$ . . . .

$\frac{1}{m+n}\left\{\frac{m}{x} + \frac{n}{2a+x}\right\} = \frac{1}{m+n} \cdot \frac{2ma + (m+n)x}{2ax + x^2} \therefore \frac{y}{p} - x$

$= \frac{(m+n)(2ax + x^2)}{2ma + (m+n)x} - x = \frac{2nax}{2ma + (m+n)x} \therefore \left(\frac{y}{p} - x\right)$

$= \frac{2na}{m+n} =$ OS.

Also, $p = \frac{y(2ma + (m+n)x)}{(m+n)(2ax+x^2)} \therefore (p) = \left(\frac{y}{x}\right) = \frac{b^{\frac{2}{m+n}}}{a^{\frac{2}{m+n}}} \ldots$

$= \tan. \angle$ S.

Cor. If $m = n = 1$, the curve is the Apollonian hyperbola.

Ex. 5. Required to draw the asymptote of the curve of Art. 7. Ex. 9.

Since AP : BP : : sin. $\angle$ B : sin. $\angle$ A, the limit of AP : BP is a ratio of equality; take then AE : EB : : n : 1, or

AE $= \frac{n}{n+1}$ AB, and the asymptote passes through E.

Also $\angle$ BEP $= \angle$ PAE $= n. \angle$ PBE $= n(2\pi - \angle$ BEP);

$\therefore \angle$ BEP $= 2\pi . \frac{n}{n+1}$, which determines the direction of the asymptote.

Ex. 6. Given the base of a plane triangle, and the difference of the angles at the base; required to draw the asymptote of the curve traced by the vertex.

Place the origin at the greatest angle; and let the base OA $= a$.

Also let $t = \tan.(\angle O - \angle A) = \frac{\frac{y}{x} - \frac{y}{a-x}}{1 + \frac{y^2}{x(a-x)}}$ (Trig. p. 35.)

$= \frac{y(a-2x)}{y^2+ax-x^2}$, $\therefore$ the curve's equation is $y^2 - \frac{a-2x}{t}y + ax - x^2 = 0$.

If this be solved as a quadratick, we have $y = \frac{a-2x}{2t}$

$\pm \frac{\sqrt{4s^2x^2 - 4as^2x + a^2}}{2t}$ where $s^2 = t^2 + 1 =$, extracting the

square root, $\frac{1}{2t}\left\{a - 2x \pm (2sx - sa - \frac{t^2a^2}{4sx} - \text{\&c.})\right\} \therefore$

the equations of the asymptotes are $y' = \frac{s-1}{t}x' - \frac{s-1}{2t}a$,

and $y' = -\frac{s+1}{t}x + \frac{s+1}{2t}a$.

Since $-\frac{s+1}{t}=-\frac{t}{s-1}$, these asymptotes intersect in the axis at right angles and at a distance from the origin $=\frac{a}{2}$.

Also $p=\frac{2x-a-\frac{2y}{t}}{2y-\frac{a-2x}{t}}$, and at the origin when $x=0$, $y=0$, $p=t$; or the curve cuts the axis at an angle = the given difference of the angles at the base.

If the origin be transferred to the point where the asymptotes intersect the axis, the equation becomes . . . .
$y^2-x^2-\frac{2xy}{t}+\frac{a^2}{4}=0$, which shows that the new origin is the *centre* of the curve (vid. 7. 6. Def. 2.); and if the axis be transferred to one of the asymptotes, according to the method (Ch. 7. Art. 18), the resulting equation will be $vw=\frac{a^2}{8}\times$ sine of given angle, or the curve is a rectangular hyperbola, which passes through O whose centre bisects AO.

Ex. 7. $y^3-axy+x^3=0$.

$p=\frac{ay-3x^2}{3y^2-ax}\therefore\frac{y}{p}=\frac{3y^3-axy}{ay-3x^2}=\frac{2axy-3x^3}{ay-3x^2}\therefore\frac{y}{p}-x$ $=\frac{axy}{ay-3x^2}=$ OT; but in the limit $y^3+x^3=0$, $\therefore y=-x$ $\therefore$ OS $=\frac{a}{3}$.

Also $(p)=-1$; take then OS $=\frac{a}{3}$, draw OD = OS perpendicular to the axis and below it, and SD is the required asymptote.

Ex. 8. $ay^4-b^3xy-cx^4=0$.

$p=\frac{b^3y+4cx^3}{4ay^3-b^3x}\therefore\frac{y}{p}=\frac{4ay^4-b^3xy}{b^3y+4cx^3}=\frac{3b^3xy+4cx^4}{b^3y+4cx^3}\therefore\frac{y}{p}-x$
$=\frac{2b^3xy}{b^3y+4cx^3}\therefore\left(\frac{y}{p}-x\right)=0$..

Also in the limit $ay^4 - cx^4 = 0 \therefore \left(\frac{y}{x}\right) = (p) = \frac{c^{\frac{1}{4}}}{a^{\frac{1}{4}}}$; or the asymptote passes through the origin inclined to the axis at $\tan.^{-1} \frac{c^{\frac{1}{4}}}{a^{\frac{1}{4}}}$.

Otherwise. $y^4 = \frac{c}{a}x^4 + \frac{b^3xy}{a}$, $\therefore$ extracting the 4th root by the binomial theorem, $y = \frac{c^{\frac{1}{4}}x}{a^{\frac{1}{4}}}\left\{1 + \frac{b^3xy}{4cx^4} - \text{\&c.}\right\}$; or the equation of the asymptote is $y' = \frac{c^{\frac{1}{4}}x'}{a^{\frac{1}{4}}}$. Vid. Ch. 4. 10. Ex. 2.

Ex. 9. $ay^3 - x^3y - ax^3 = 0$.

$p = \frac{3x^2y + 3ax^2}{3ay^2 - x^3} \therefore \frac{y}{p} = \frac{3ay^3 - x^3y}{3x^2y + 3ax^2} = \frac{2x^3y + 3ax^3}{3x^2y + 3ax^2} \therefore \frac{y}{p} - x = \frac{-x^3y}{3x^2y + 3ax^2} \therefore \left(\frac{y}{p} - x\right) = -\infty$; or there is not a rectilinear asymptote which cuts the axis x at a finite distance. But since (Ch. 4. 9. Ex. 3.) $y = -a - \frac{a^4}{x^3} - \frac{3a^7}{x^6} - \text{\&c.}$ there is an asymptote parallel to the axis and below it which cuts Y at a distance from the origin $= a$.

The curve has also a curvilinear asymptote whose equation is $y' = \frac{x'^{\frac{3}{2}}}{a^{\frac{1}{2}}} + \frac{a}{2}$ (4. 10. Ex. 1).

Ex. 10. $y^3 - 2xy^2 + x^2y - a^3 = 0$.

The values of y are (4. 42. Ex. 4),

$$\left.\begin{aligned} y &= x \pm \frac{a^{\frac{3}{2}}}{x^{\frac{1}{2}}} - \frac{a^3}{2x^2} \pm \text{\&c.} \\ y &= \frac{a^3}{x^2} + \frac{2a^6}{x^5} - \text{\&c.} \end{aligned}\right\}.$$

Hence there is a rectilinear asymptote which passes through the origin cutting the axis at an $\angle\, 45°$, and since

when $x = \infty$, one value of y is indefinitely small, the axis X is also an asymptote of the curve.

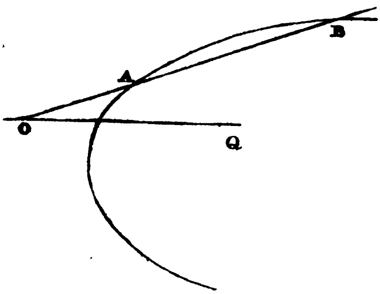
Curves may have asymptotes at other points than those which are determined by the above method: take for example the equation $y = \dfrac{\pm a\sqrt{ax - x^2}}{x}$; when $x = 0$, $y = \infty$, which shows that the curve has an infinite ordinate or asymptote passing through the origin.

Generally, if the curve's equation is $y = \dfrac{P}{Q}$ where P and Q are functions of x, there are asymptotes at distances from the origin equal to the roots of the equation $Q = 0$.

For additional examples of asymptotes to curves, and for further information on the subject, vid. Stirling's Commentary on Sir I. Newton's *Enumeratio Linearum 3tii ordinis.*

14. *Required to investigate the direction of the infinite branches of a curve.*

The direction of the axis of the infinite branch is known, if we can find the direction of that line, which drawn from any point whatever in it, and produced indefinitely both ways, cuts the branch in only one point.



Let OAB be the given axis of the abscissæ, then if there is an infinite branch, the direction of whose axis is OQ, as OAB revolves through the $\angle$ BOQ, AB increases and becomes infinite. Also the ordinate at B is always $= 0$; whence the characteristick property of an infinite branch is that the direction of the axis of the abscissæ may be so changed that the ordinate being $= 0$, the abscissa may be infinite.

Let x and y, v and w be the co-ordinates on the axes OB, OQ respectively, $\theta = \angle O$; and let the curve's equation be $Ay^n + By^{n-1}x + \ldots . Ex^n + Fy^{n-1} + Gy^{n-2}x + \ldots . + N = 0.$

In this equation substitute $x = pv$ and $y = qv$, where $p = \cos.\theta$ and $q = \sin.\theta$ (7. 18.), and it becomes . . . . .

$(\text{A}q^n+\text{B}q^{n-1}p+\ldots.+\text{E}p^n)v^n+(\text{F}q^{n-1}+\text{G}q^{n-2}p+\ldots.)v^{n-1}$ $+\ldots.+\text{N}=0$. Transform this into one whose roots are its reciprocals, and the proposed curve has an infinite branch when one root of the transformed equation $=0$, *i. e.* when $\text{A}q^n+\text{B}q^{n-1}p+\ldots.+\text{E}p^n=0$, or dividing by p^n and substituting $t=\frac{q}{p}$ when $\text{A}t^n+\text{B}t^{n-1}+\ldots.+\text{E}=0$; the roots of which equation will indicate the number and the direction of the axes of the infinite branches.

If all the values of t are impossible, the curve does not admit of an infinite branch. It may have an asymptote at *right angles* to the axis of the abscissæ.

The direction OQ indicates *two* infinite branches, one on each side of the axis; for assigning a determinate value to one of the co-ordinates, and supposing the other to be infinite, its positive and negative values will each produce an infinite branch.

The equation obtained in this article for determining the direction of OQ may be deduced immediately from the doctrine of infinitesimals. For the curve's equation, when x and y are infinite, takes the form of $\text{A}y^n+\text{B}y^{n-1}x+\ldots.$ $+\text{E}x^n=0$; or dividing by x and substituting $t=\frac{y}{x}$, $\text{A}t^n+\text{B}t^{n-1}+\ldots.+\text{E}=0$ is an equation whose roots will show whether any part of the curve is at an infinite distance.

Cor. 1. A curve whose equation is of an *odd* number of dimensions, will have, at least, one infinite branch on each side of the axis (Alg. 278).

Cor. 2. If the curve's equation is of n dimensions, it cannot have more than $2n$ infinite branches.

Cor. 3. The ultimate direction of the infinite branch is found by calculating the value of $\frac{dy}{dx}$ when x and y are infinite.

Ex. $ay^2+(b+cx)y+ex^2+fx+g=0$.

This equation arranged according to its dimensions is $ay^2+cxy+ex^2+by+fx+g=0$; whence the equation for determining the infinite branches is $at^2+ct+e=0$,

or $t=\frac{-c\pm\sqrt{c^2-4ae}}{2a}$.

When $4ae$ is greater than c^2 there is no infinite branch; when $4ae = c^2$, t has but one value, or there is an infinite branch on each side of the axis which are ultimately parallel: when $4ae$ is less than c^2, the curve has four infinite branches.

It is called an hyperbolick or a parabolick branch according as it has or has not an asymptote.

On the subject of the classification of curves, I must refer the reader to the Enumeratio Linearum 3tii ordinis.

Tangent planes.

15. *Def.* The *tangent plane* at any point of a curve surface is that part of a plane drawn touching the surface which is intercepted between the rectangular axes.

16. *Def.* The *subtangent* is that part of the axis of the solid which is intercepted by the tangent plane.

The origin is here supposed to be at the centre of the solid. The axis Z is generally taken as the axis of the solid; in which case OC is the subtangent (fig. 7. 47). The lines OA, OB are called the subtangents on the axes X and Y.

17. *Def.* The *normal* of any point of a curve surface is that part of a line drawn at right angles to the surface which is intercepted by either of the co-ordinate planes XZ or YZ.

The normal of the tangent plane is a line drawn from the origin perpendicular on the plane.

18. *The square of the tangent plane is equal to the sum of the squares of its orthographick projections on the three co-ordinate planes.*

For let P be the area of the plane;

x, y, z, its projections on YZ, XZ, XY.

Also, let α, β, γ be the angles which the plane makes with these planes respectively.

Then $x = \text{P}\cos.\alpha$, $y = \text{P}\cos.\beta$, $z = \text{P}\cos.\gamma$; therefore $x^2 + y^2 + z^2 = \text{P}^2(\cos.^2\alpha + \cos.^2\beta + \cos.^2\gamma) = \text{P}^2$.

Cor. $\text{P} = \sqrt{x^2 + y^2 + z^2} = \frac{1}{2}\sqrt{\text{A}^2\text{B}^2 + \text{A}^2\text{C}^2 + \text{B}^2\text{C}^2}$, where $\text{OA} = \text{A}$, $\text{OB} = \text{B}$, $\text{OC} = \text{C}$.

19. *Required to draw a plane touching a given surface at a given point.*

Let P be the point of contact, whose co-ordinates are x, y, z; $z' = mx' + ny' + c$ the required plane; then, since it passes through (x, y, z), we have $z' - z = m(x' - x) + n(y' - y)$. (α).

And to determine m and n; suppose a plane to be drawn through P parallel to XZ: its intersections with the surface and with the tangent plane are a curve and a right line which touch at the point P; project these intersections on XZ which will not affect the values of x and z, and we shall have a right line whose equation is of the form $z' = mx' + \mu$ a tangent to a curve whose equation is of the form $z = \mathrm{F}x$, for y' and y are constant; whence (Art. 3.) $m = \frac{dz}{dx}$.

Similarly it may be shown that $n = \frac{dz}{dy}$; or (α) becomes

$$z' - z = \frac{dz}{dx}(x' - x) + \frac{dz}{dy}(y' - y).$$

Cor. Substitute $p = \frac{dz}{dx}$, and $q = \frac{dz}{dy}$, and the equation becomes $z' - z = p(x' - x) + q(y' - y)$.

20. *Required the angles which the tangent plane makes with the co-ordinate planes.*

Let γ, β, α be the angles which it makes with XY, XZ, YZ respectively; then, the plane passing through the point (x, y, z), its equation may be put under the form $x\cos.\alpha + y\cos.\beta + z\cos.\gamma = \mathrm{R}$ (7. 53. Cor. 3.); wherefore $p = \frac{dz}{dx} = -\frac{\cos.\alpha}{\cos.\gamma}$, $q = \frac{dz}{dy} = -\frac{\cos.\beta}{\cos.\gamma}$; and consequently $1 + p^2 + q^2 = 1 + \frac{\cos.^2\alpha}{\cos.^2\gamma} + \frac{\cos.^2\beta}{\cos.^2\gamma} = \frac{1}{\cos.^2\gamma}$, or . . . . .

$$\cos.\gamma = \frac{1}{\sqrt{1 + p^2 + q^2}}; \text{ and } \cos.\beta = q \times -\cos.\gamma = \frac{-q}{\sqrt{1 + p^2 + q^2}};$$

and $\cos.\alpha = \frac{-p}{\sqrt{1 + p^2 + q^2}}$.

21. *Required to draw a normal line from a given point of a curve surface.*

Let P be the given point, PG the required normal; at P suppose the tangent plane to be drawn, and let its equation be $z' = mx' + ny' + c$: then PG is also a normal to this plane, and consequently its equations in space, or its projections upon XZ and YZ are respectively at right angles to the lines $z' = mx' + c$ and $z' = ny' + c$ (7. 51 and 64.); or the

equations in space are of the form $\left.\begin{array}{l} z'' = -\dfrac{1}{m}x'' + \mu \\ z'' = -\dfrac{1}{n}y'' + \nu \end{array}\right\}$,

(7. 11.); and since the line passes through (x, y, z), the required equations are $\left.\begin{array}{l} z'' - z + \dfrac{1}{m}(x'' - x) = 0 \\ z'' - z + \dfrac{1}{n}(y'' - y) = 0 \end{array}\right\}$

or, $\left.\begin{array}{l} x' - x + p(z' - z) = 0 \ (1) \\ y' - y + q(z' - z) = 0 \ (2) \end{array}\right\}$ where x', y', z' are the co-ordinates of any point of the normal.

Cor. 1. The third equation in space of the normal is $q(x' - x) - p(y' - y) = 0$ (3).

Cor. 2. Hence to find the point where the normal meets either of the co-ordinate planes as XZ; in equation (2) suppose $y' = 0$, which will give the required value of z'; substitute this in equation (1), and the co-ordinates of the required point will be found.

Similarly by means of the equation (3), the co-ordinates of the points where the normal meets YZ and XY may be found.

Cor. 3. If α, β, γ are the angles which the normal makes with the axes X, Y, Z; $\cos.\alpha = \dfrac{-p}{\sqrt{1+p^2+q^2}}$, ...

$\cos.\beta = \dfrac{-q}{\sqrt{1+p^2+q^2}}$, and $\cos.\gamma = \dfrac{1}{\sqrt{1+p^2+q^2}}$.

Cor. 4. The length of the normal PG $= z\sqrt{1+p^2+q^2}$, if G is the point where the normal meets XY.

For $x' - x = pz$, $y' - y = qz$; therefore PG, which $= \sqrt{(x' - x)^2 + (y' - y)^2 + z^2} = z\sqrt{1 + p^2 + q^2}$.

Similarly it may be shown that the normals terminating in XZ and YZ are $= \dfrac{y}{q}\sqrt{1 + p^2 + q^2}$ and $\dfrac{x}{p}\sqrt{1 + p^2 + q^2}$ respectively.

22. *Required to draw a normal line from a given point to a given curve surface.*

If x, y, z are the co-ordinates of that point where the normal meets the surface, the equations of the normal are

$$\left.\begin{array}{l} x' - x + p(z' - z) = 0 \\ y' - y + q(z' - z) = 0 \end{array}\right\}.$$

Let x'', y'', z'' be the co-ordinates of the given point, then since the normal passes through it, we have by elimination $\left.\begin{array}{l} x' - x'' + p(z' - z'') = 0 \\ y' - y'' + q(z' - z'') = 0 \end{array}\right\}$, which are the required equations, where p and q are to be calculated from the equation of the surface.

23. *Examples.*

Ex. 1. $z = e\sqrt{x^2 + y^2}$; which is a conical surface whose vertex is at the origin, and the axis is z. (7. 45).

$$\left.\begin{array}{l} p = \dfrac{ex}{\sqrt{x^2+y^2}} = \dfrac{e^2x}{z} \\ q = \dfrac{ey}{\sqrt{x^2+y^2}} = \dfrac{e^2y}{z} \end{array}\right\}.$$

(1.) Required the equation of the tangent plane.

$$z' - z = \frac{e^2x}{z}(x' - x) + \frac{e^2y}{z}(y' - y), \text{ or } \frac{zz'}{e^2} = xx' + yy'.$$

To find where the plane cuts the axis, suppose $x' = 0$ and $y' = 0$; then $z' = 0$; or the tangent plane always passes through the vertex.

This is the characteristick property of *conical* surfaces.

(2.) Required the angle which the tangent plane makes with the axis $1 + p^2 + q^2 = 1 + e^2$, and $\therefore$ sine of required $\angle$, which $= \cos.\gamma = \dfrac{1}{\sqrt{1+p^2+q^2}} = \dfrac{1}{\sqrt{1+e^2}}$, or the angle is invariable.

(3.) Required the point where the normal meets the plane xz.

The equations of the normal are $\left.\begin{array}{l} x' - x + \dfrac{e^2x}{z}(z' - z) = 0 \\ y' - y + \dfrac{e^2y}{z}(z' - z) = 0 \end{array}\right\}$

in which, if $y' = 0$, we have $\dfrac{e^2}{z}(z' - z) = 1$; and by substitution, $x' - x + x = 0$, or $x' = 0$; and $z' = \dfrac{1+e^2}{e^2}z$; or the normal meets the axis of the cone at a distance from the

origin $= \frac{1+e^2}{e^2} z = \frac{z}{\cos.^2\alpha}$, if α is $\frac{1}{2}$ the vertical angle of the cone.

(4.) The length of the normal $= \sqrt{1+e^2}\, z$.

Ex. 2. $\frac{x}{a} + \frac{y}{b} = \frac{\sqrt{a^2 - z^2}}{a}$; which is the surface of an irregular groin (Ch. 7. Art. 46. Ex. 3.).

$$\left.\begin{aligned} \text{Here } p &= -\frac{\sqrt{a^2-z^2}}{z} \\ q &= -\frac{a}{b}\frac{\sqrt{a^2-z^2}}{z} \end{aligned}\right\}.$$

(1.) The equation of the tangent plane is $\frac{x'}{a} + \frac{y'}{b} \ldots$

$$+\frac{zz'}{a\sqrt{a^2-z^2}} = \frac{a}{\sqrt{a^2-z^2}}.$$

It appears from this equation, that so long as z is constant, the position of the tangent plane is fixed; that it cuts the axis at a distance from the origin $= \frac{a^2}{z}$, and at an angle the sine of which $= \frac{bz}{a\sqrt{a^2+b^2-z^2}}$.

(2.) Required the point where the normal meets XZ.

The equations of the normal are

$$\left.\begin{aligned} x' - x - \frac{\sqrt{a^2-z^2}}{z}(z' - z) &= 0 \\ y' - y - \frac{a}{b}\frac{\sqrt{a^2-z^2}}{z}(z' - z) &= 0 \end{aligned}\right\}, \text{ from which, if } y' = 0,$$

we have by substitution $x' = \frac{ax - by}{a}$ and $z' = z - \frac{byz}{a\sqrt{a^2-z^2}}$ for the required co-ordinates.

To find the point where the normal meets YZ; suppose $x' = 0$; and we have $y' = \frac{by - ax}{b}$ and $z' = z - \frac{xz}{\sqrt{a^2-z^2}}$ for the co-ordinates of G.

The co-ordinates of the point where the normal meets XY are $\left.\begin{aligned} x' &= x - \sqrt{a^2 - z^2} \\ y' &= y - \frac{a}{b}\sqrt{a^2 - z^2} \end{aligned}\right\}.$

(3.) The length of the normal terminating in . . . . .
$\text{XY} = \frac{az}{b}\sqrt{\frac{b^2+z^2}{a^2-z^2}}.$

Ex. 3. $\frac{x}{\sqrt{a^2-z^2}} + \frac{y}{b} = 1$; the convex wedge, called by Wallis the *cono-cuneus.*

$$\left.\begin{aligned} \frac{1}{\sqrt{a^2-z^2}} + \frac{pxz}{(a^2-z^2)^{\frac{3}{2}}} = 0 \therefore p = -\frac{a^2-z^2}{xz} \\ \frac{qxz}{(a^2-z^2)^{\frac{3}{2}}} + \frac{1}{b} = 0 \therefore q = -\frac{(a^2-z^2)^{\frac{3}{2}}}{bxz} \end{aligned}\right\}.$$

(1.) The equation of the tangent plane is $bxzz' + b(a^2-z^2)x' + (a^2-z^2)^{\frac{3}{2}}y' = a^2bx + (a^2-z^2)^{\frac{3}{2}}y.$

(2.) Required the point where the normal meets XZ.
The equations of the normal are

$$\left.\begin{aligned} x' - x - \frac{a^2-z^2}{xz}(z'-z) = 0 \\ y' - y - \frac{(a^2-z^2)^{\frac{3}{2}}}{bxz}(z'-z) = 0 \end{aligned}\right\}$$ whence, if $y' = 0$, by substitution we have $x' = x - \frac{by}{\sqrt{a^2-z^2}}$ and $z' = z - \frac{bxyz}{(a^2-z^2)^{\frac{3}{2}}}.$

Ex. 4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; the equation of an ellipsoid.

Here $\left.\begin{aligned} p &= -\frac{c^2x}{a^2z} \\ q &= -\frac{c^2y}{b^2z} \end{aligned}\right\}.$

(1.) The equation of the tangent plane is

$$z' - z = -\frac{c^2x}{a^2z}(x'-x) - \frac{c^2y}{b^2z}(y'-y), \text{ or } \frac{xx'}{a^2} + \frac{yy'}{b^2} \ldots\ldots + \frac{zz'}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Hence the tangent plane cuts the axes X, Y, Z at distances from the origin $= \frac{a^2}{x}, \frac{b^2}{y}, \frac{c^2}{z}$ respectively, and at angles whose sines are $\frac{xk}{a^2}, \frac{yk}{b^2}, \frac{zk}{c^2}$, (7. 52) where . . . . . .

$$k = \frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}.$$

Cor. 1. If $c = \infty$, the ellipsoid becomes an elliptick cylinder, whose base is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, and the equation of whose tangent plane is $\frac{xx'}{a^2}+\frac{yy'}{b^2} = 1$.

Cor. 2. The tangent plane of the cylinder is parallel to the axis.

This is the characteristick property of *cylindrical* surfaces *.

(2.) The area of the tangent plane $= \frac{a^2b^2c^2}{2xyz} \times \frac{1}{k}$.

(3.) The content of a pyramid formed by the tangent plane and the co-ordinate planes $= \frac{1}{6} \cdot \frac{a^2b^2c^2}{xyz}$.

For a pyramid = the base × $\frac{1}{3}$ altitude (Ch. 9. 7. Ex. 4.).

(4.) The equations of the normal are

$$\left.\begin{aligned} x'-x &= \frac{c^2x}{a^2z}(z'-z) \\ y'-y &= \frac{c^2y}{b^2z}(z'-z) \end{aligned}\right\}$$

Cor. 1. The co-ordinates of G in XZ are $x' = \frac{a^2-b^2}{a^2}x$, . . . $z' = \frac{c^2-b^2}{c^2}z$.

Cor. 2. If a and c be both greater than b, G lies in the first quadrant.

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* The general properties of conical and cylindrical surfaces, as well as the evolutes and the singular points of curve surfaces will be considered in the second volume.

Cor. 3. If $a = b$, the solid becomes a spheroid, and $x' = 0$; or the normal intersects the axis z.

Cor. 4. If $a = b = c$, the solid is a sphere, and $x' = 0$, $z' = 0$; or the normal passes through the origin.

(5.) The length of the normal $= -\dfrac{c^2}{k}$.

(6.) The normal of the tangent plane $= k$.

Ex. 5. $z = \dfrac{x^2}{a} + \dfrac{y^2}{b}$; the elliptick paraboloid.

(1.) The equation of its tangent plane is

$$z' - z = \frac{2x}{a}(x' - x) + \frac{2y}{b}(y' - y), \text{ or } 1 = \frac{2xx'}{az} + \frac{2yy'}{bz} - \frac{z'}{z}.$$

(2.) The subtangent $= - z$; and the subtangents on the axes X and Y are $\dfrac{az}{2x}$ and $\dfrac{bz}{2y}$.

(3.) The equations of the normal are

$$\left.\begin{aligned} x' - x + \frac{2x}{a}(z' - z) &= 0 \\ y' - y + \frac{2y}{b}(z' - z) &= 0 \end{aligned}\right\}.$$

(4.) The co-ordinates of the point where the normal meets XZ are $z' = \dfrac{b}{2} + z$ and $x' = \dfrac{a-b}{a}x$.

If $a = b$, $x' = 0$, or the normal intersects the axis of the solid.

(5.) The area of the tangent plane

$$= \frac{abz^2}{4xy}\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{1}{4}}.$$

Ex. 6. $\dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$; the elliptick hyperboloid.

(1.) The tangent plane is $z' - z = \dfrac{c^2x}{a^2z}(x' - x) \ldots\ldots$
$+ \dfrac{c^2y}{b^2z}(y' - y)$, or $\dfrac{zz'}{c^2} - \dfrac{xx'}{a^2} - \dfrac{yy'}{b^2} = 1$.

(2.) The subtangent $= \dfrac{c^2}{z}$; and the subtangents on the axes X and Y are $-\dfrac{a^2}{x}$ and $-\dfrac{b^2}{y}$.

(3.) The area of the tangent plane $= \dfrac{a^2b^2c^2}{2xyz}k$, where . . .

$$k = \sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}.$$

(4.) The content of the pyramid formed by the tangent plane and the co-ordinate planes $= \dfrac{1}{6} \cdot \dfrac{a^2b^2c^2}{xyz}$.

(5.) The co-ordinates of the point when the normal meets XZ are $\left.\begin{aligned} x' &= x - \frac{b^2x}{a^2} \\ z' &= \frac{b^2+c^2}{c^2}z \end{aligned}\right\}.$

Ex. 7. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$; the elliptick transverse hyperboloid.

(1.) The tangent plane is $\dfrac{xx'}{a^2} + \dfrac{yy'}{b^2} - \dfrac{zz'}{c^2} = 1$.

(2.) The subtangent $= -\dfrac{c^2}{z}$.

(3.) The area of the tangent plane $= \dfrac{a^2b^2c^2}{2xyz}k$.

(4.) The content of the pyramid formed by the tangent plane and the co-ordinate planes $= -\dfrac{1}{6} \dfrac{a^2b^2c^2}{xyz}$.

(5.) The co-ordinates of the point where the normal meets XZ are $\left.\begin{aligned} x' &= x - \frac{b^2x}{a^2} \\ z' &= z + \frac{b^2}{c^2}z \end{aligned}\right\}.$

Ex. 8. $z = \dfrac{y^2}{b} - \dfrac{x^2}{a}$.

The co-ordinate sections are $z = \dfrac{y^2}{b}$; $-z = \dfrac{x^2}{a}$; and $y = \pm \dfrac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}}x$. Or the sections made by the planes XZ, YZ

are two parabolas; the one below, and the other above the plane XY, each having its vertex at O. The section in the plane XY consists of two right lines which are drawn from O on different sides of XZ inclined at the same angle to X, viz. $\tan.^{-1}\sqrt{\frac{b}{a}}$.

Also all sections parallel to XY are hyperbolas, and hence this solid is called the *hyperbolick paraboloid.*

It may be supposed to be generated by an hyperbola whose centre moves along the axis Z, whose plane is parallel to XY, and whose semi-major and minor axes are the ordinates of the parabolas in the planes YZ and XZ respectively.

When z is negative, the equation of the generating section becomes $\frac{x^2}{az}-\frac{y^2}{bz}=1$, which shows that when the generating section is below the plane of the paper, the semi-axis major of the hyperbola is the ordinate of the *lower* parabola.

The equation of the tangent plane is $-\frac{2xx'}{az}+\frac{2yy'}{bz}-\frac{z'}{z}=1$, from which the principal properties of the solid may be deduced as in Ex. 5.

Ex. 9. One of the angular points of a paralleleliped moves so that the content of the solid may be invariable; required the tangent plane at any point of the surface which it traces.

In fig. 7. 38 let P be the variable point, then the equation of the surface is $xyz=a^3$; and $p=-\frac{a^3}{x^2y}=-\frac{z}{x}$, and ... $q=-\frac{z}{y}$; and the equation of the tangent plane is $z'-z=-\frac{z}{x}(x'-x)-\frac{z}{y}(y'-y)$ or $\frac{z'}{z}+\frac{x'}{x}+\frac{y'}{y}=3$.

(2.) The subtangents are $3z$, $3x$, $3y$.

(3.) The area of the tangent plane $=\frac{9a^3}{2}\sqrt{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}}$ $=\frac{9a^3}{2k}$.

(4.) The content of the pyramid formed by the co-ordinate planes $= \frac{1}{6} \cdot 27xyz = \frac{9a^3}{2}$, a constant quantity.

(5.) The length of the normal terminating in XY $= z\sqrt{1+p^2+q^2} = -z^2\sqrt{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}} = -\frac{z^2}{k}$.

(6.) Required the point where the normal meets XZ.

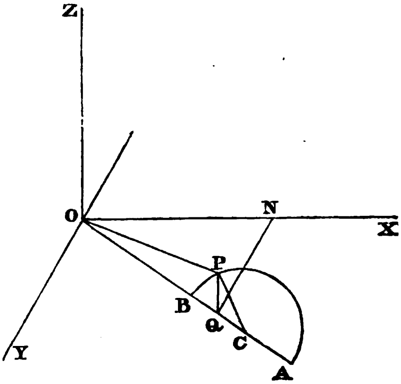
$$\left.\begin{array}{r} x'-x-\frac{z}{x}(z'-z)=0 \\ -y-\frac{z}{y}(z'-z)=0 \end{array}\right\} \therefore\ z'=\frac{z^2-y^2}{z} \text{ and } x'=\frac{x^2-y^2}{x}.$$

(7.) The normal of the tangent plane $= 3k$.

Ex. 10. A solid annulus in the form of the ring of an anchor; it is generated by a circle revolving round a fixed axis in a plane perpendicular to itself.

Let OA revolve round O in the plane XY; and let the semi-circle BPA, which is perpendicular to XY, trace half the solid.

Draw the co-ordinates of any point P, in the surface ON, NQ, QP;



$OC = a$ | where C is the centre of the circle.
$CB = r$ | Join OP, CP.

Since the circle is perpendicular to XY, PQ is an ordinate to the abscissa CQ; and we have $OP^2 = OC^2 + CP^2 \pm 2OC.CQ$ (Eu. 2. 12.); or $x^2+y^2+z^2 \pm 2a\sqrt{r^2-z^2} = a^2+r^2$ is the equation of the solid.

A section made by a plane parallel to XY is of the form of two concentrick circles; and consequently the solid may be supposed to be generated by a superficial annulus of variable breadth moving along the axis Z in a plane perpendicular to it; the breadth of the annulus being equal to twice the abscissa of the circle APB.

If the circle is inclined to the plane in which it revolves at an $\angle\, a$, its ordinate corresponding to $z = \dfrac{z}{\sin. a}$ and CQ $= \dfrac{\sqrt{r^2 \sin.^2 a - z^2}}{\sin. a}$ and the equation becomes $x^2 + y^2 + z^2 \pm \dfrac{2a}{\sin. a}\sqrt{r^2 \sin.^2 a - z^2} = a^2 + r^2$.

If APB be any other known curve, the equation of the solid may be found.

(1.) The equation of the tangent plane is $azz' - \sqrt{r^2 - z^2}\,(xx' + yy' + zz') = az^2 - \sqrt{r^2 - z^2}\,(x^2 + y^2 + z^2)$ where a may be either positive or negative.

(2.) The normal passes through the axis.

(3.) Required the length of the normal.

$$p = \frac{x\sqrt{r^2 - z^2}}{z(a - \sqrt{r^2 - z^2})}, \text{ and } q = \frac{y\sqrt{r^2 - z^2}}{z(a - \sqrt{r^2 - z^2})},$$

$$\therefore\ 1 + p^2 + q^2 = 1 + \frac{r^2 - z^2}{z^2(a - \sqrt{r^2 - z^2})^2}(x^2 + y^2)$$

$$= \frac{z^2(a^2 - 2a\sqrt{r^2 - z^2}) + (r^2 - z^2)(x^2 + y^2 + z^2)}{z^2(a - \sqrt{r^2 - z^2})^2}$$

$$= \frac{r^2(x^2 + y^2)}{z^2(a - \sqrt{r^2 - z^2})^2}, \ \therefore \text{PG, which} = \frac{x}{p}\sqrt{1 + p^2 + q^2},$$

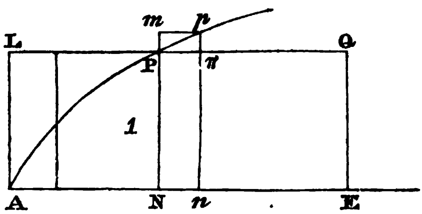
$$= -r\sqrt{\frac{x^2 + y^2}{r^2 - z^2}}.$$

CHAPTER IX.

The Quadrature of Areas; the Rectification of Curves; and the Cubature of Solids.

1. *Required to find the area of a known curve whose co-ordinates are rectangular.*

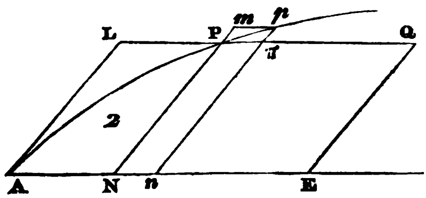
Let the origin be at A the vertex of the curve: ANP the area $= u$.



AN $= x$ | PN $= y$ | In the axis of the abscissæ take NE finite and constant to represent the fluxion of AN; complete the parallelograms NPLA, NPQE; and draw $p\pi n$ parallel and near to PN; and complete the parallelogram Nnpm.

Suppose the parallelogram NL and the area APN to be generated, the one by a constant and the other by a variable ordinate; then at N the increment of APN : the cotemporary increment of LN :: NPpn : Nπ; and to obtain the limit of this ratio, diminish Nn indefinitely; then since mn : Pn :: Nm : NP ultimately :: 1 : 1 (1. 39); therefore, à fortiori, NPpn : Nπ ultimately :: 1 : 1, and consequently the fluxion of APN = the fluxion of LN = PN $\times$ the fluxion of AN = PN $\times$ NE = NQ; or $du = ydx$ and $u = \int ydx$.

Hence, to find the area from the curve's equation, calculate ydx in terms either of x or of y, and its fluent properly corrected will give the area in terms of one of its co-ordinates.



Cor. 1. If the co-ordinates are not rectangular, but inclined at an angle whose trigonometrical sine $= s$, it may be shown, as in the article, that $du = s \times ydx$, or $u = s\int ydx$.

Cor. 2. The fluxion of an area generated by a line at right angles to the axis is as the generating line and its velocity jointly. And as a curve line moving in a direction perpendicular to its plane must generate an area of the same magnitude as a right line of equal length; the fluxion of a *surface* generated by a curve is as the curve and its velocity jointly.

Cor. 3. The fluxion of the outer area APL, supposing AL to be the axis and PL the generating ordinate, $=$ PL $\times$ d.AL $= xdy$; hence, when both sides of the parallelogram vary, its fluxion $= ydx + xdy$.

Cor. 4. Since ydx changes its sign when either y or x becomes negative, the area is positive or negative according as the co-ordinates have the same or different signs. It is positive in the 1st and 3d quadrants and negative in the 2d and 4th.

2. *The quadrature of areas by means of Taylor's Theorem.*

Let $y = fx$ be the curve's equation, then since the area u is a function of the co-ordinates which enclose it, it may be considered as an explicit function of x: and therefore by Taylor's Theorem we have

$$\left.\begin{aligned} \text{NP}\pi n &= yh \\ \text{NP}pn &= \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \&\text{c.} \\ \text{N}mpn &= h\left\{y + \frac{dy}{dx}\frac{h}{1} + \&\text{c.}\right\} \end{aligned}\right\}$$

of which the second is always greater than the first and less than the third; consequently (3.3)

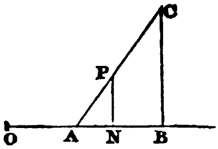
$$\frac{du}{dx}h = yh \text{ or } du = ydx \text{ and } u = \int ydx.$$

In the following examples, the origin is either at the vertex or at the centre of the curve, and the co-ordinates are always supposed rectangular unless the contrary is expressed.

If the origin is changed to some other known point in the axis, this will only affect the correction of the fluent.

3. *Examples.*

Ex. 1. A right angled triangle.



AB $= a$ BC $= b$ AN $= x$ PN $= y$	$\Delta^s\ x : y :: a : b\ \therefore\ y = \frac{b}{a}x$ $ydx = \frac{b}{a}xdx\ \therefore$ ANP $=$

$\frac{b}{a} \cdot \frac{x^2}{2}$, for $c = 0$; and to obtain the whole area, make $x = a$, then ABC $= \frac{b}{a} \cdot \frac{a^2}{2} = \frac{ab}{2}$.

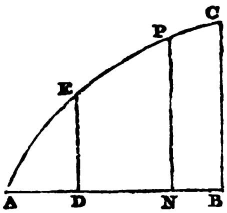
Ex. 2. An Apollonian parabola.

$$y^2 = ax\ \therefore\ dx = \frac{1}{a} \cdot 2ydy\ \therefore\ ydx = \frac{2}{a}\ y^2dy = du\ \therefore$$

$u = \frac{2}{3}\frac{y^3}{a} = \frac{2}{3}xy$, and the whole area $= \frac{2}{3}$ circumscribing parallelogram.

When the co-ordinates are not rectangular, the area of the parabola $= \frac{2}{3}$ circumscribing parallelogram.

Ex. 3. Required to find the area intercepted between two known ordinates BC and DE.



BC $= b$ DE $= c$	Let NP be the generating ordinate; then as before

$d.$DEPN $= d.$APN $= \frac{2}{a} \cdot y^2dy$;

whence,

$$\left.\begin{aligned} \text{DEPN} &= \frac{2}{3}\frac{y^3}{a} + \text{C} \\ \text{o} &= \frac{2}{3}\frac{c^3}{a} + \text{C} \end{aligned}\right\}\therefore\ \text{DEPN} = \frac{2}{3a} \cdot (y^3 - c^3)$$

and DECB $= \frac{2}{3a} \cdot (b^3 - c^3)$.

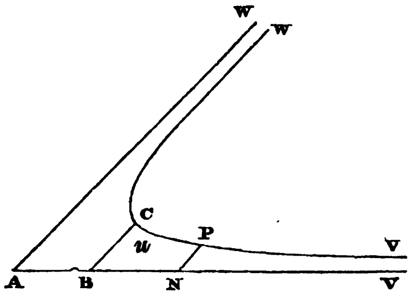
Ex. 4. $a^{n-1}y = x^n$ is the general equation of the parabola when n is positive, and of the hyperbola between the asymptotes when n is negative.

It may be shown as in the preceding example, that when $a^{n-1}y = x^n$, $u = \frac{1}{n+1}. xy$; hence, all parabolick areas are *quadrable*, or can be expressed in finite terms of the co-ordinates.

Cor. When $n = 1$, the parabola becomes a right line and $u = \frac{1}{2} xy$.

Ex. 5. Required the area of an hyperbola between the asymptotes, where the co-ordinates are not rectangular.

The equation is $y = \frac{a^{n+1}}{x^n}$;



AB $= b$ AN $= x$ NP $= y$ S $=$ sin. $\angle$ A	$\therefore du = s \times ydx = sa^{n+1}.\frac{dx}{x^n}$ . . . . $\therefore \; u = \frac{sa^{n+1}}{1-n} \times x^{1-n} + \text{C}$

$$\text{and } o = \frac{sa^{n+1}}{1-n} \times b^{1-n} + \text{C}$$

$$\therefore \; u = \frac{sa^{n+1}}{1-n}.\left\{x^{1-n} - b^{1-n}\right\}.$$

(1.) If n be less than 1, when $x = \infty$, $u = \infty$ or BVVC is infinite.

(2.) If n be greater than 1, u becomes $\frac{sa^{n+1}}{n-1}\left\{\frac{1}{b^{n-1}} - \frac{1}{x^{n-1}}\right\}$, or the area BVVC, though infinite in extent, is finite in quantity.

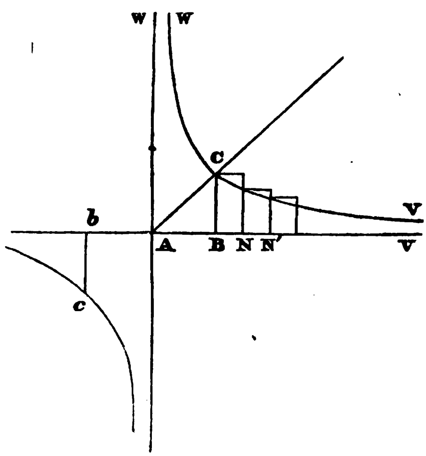
To obtain the value of the area BAWWC, make $x = 0$.

(3.) First, suppose n less than 1, then $(u) = \frac{sa^{n+1}}{1-n} \times -b^{1-n}$, and as the negative sign only shows that the area is to be reckoned on the contrary side of BC, we have BAWWC $= \frac{sa^{n+1}}{1-n} \times b^{1-n}$, a finite quantity.

(4.) Let n be greater than 1, $\therefore u = \frac{sa^{n+1}}{n-1}\left\{\frac{1}{b^{n-1}} - \frac{1}{x^{n-1}}\right\}$ = (when $x = 0$) ∞, or BAWWC is in this case infinite.

Hence, whether n be greater or less than unity, the whole area between the asymptotes is infinite.

When $n = 1$, the curve is the Apollonian hyperbola whose equation is $xy = a^2$; but if we substitute 1 for n in the value of u, we have $u = sa^2.\frac{0}{0}$, which shows that $\int\frac{dx}{x^n}$ cannot be found in this case by the common rules of integration.—Vid. Ch. 5. 15. Ex. 4., from which it appears that $(u) = sa^2.\ l\frac{x}{b}$.



First, suppose the asymptotes to be rectangular, or the hyperbola to be equilateral.
AB = BC = $a = 1$
BN = x, NP = y*
then AN = $1 + x$, and reckoning the area from BC, we have BCPN $= \int y dx = \int\frac{dx}{1+x} = l.$ AN; or the hyperbolick areas, when the hyperbola is rectangular, are the Naperian logarithms of their abscissæ.

If we take abscissæ AN, AN$'$, AN$''$, &c. in geometrick progression, the areas BCPN, BCP$'$N, BCP$''$N$''$, &c. will increase in arithmetick progression.

If the hyperbola is not rectangular, the hyperbolick areas are the logarithms of their abscissæ in some other system than the Naperian; the sine of the angle between the asymptotes being equal to the modulus of the system.

Thus, to find the $\angle$ A so that the hyperbolick areas may

* P and P$'$ are omitted in the diagram.

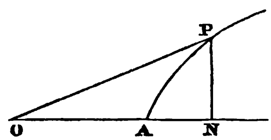
be the *common* logarithms of their abscissæ, we have sin. $\angle$ A $= .4342945 \ldots$ which is Briggs' modulus, and consequently $\angle$ A nearly $= 25^\circ\ 55'\ 16''\ 10'''$.

Cor. 1. By means of this example, we can determine when the sum of the series $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} +$ &c., ad inf. is finite and when infinite.

Suppose the co-ordinates rectangular, and take AB $= 1$ $=$ BN $=$ NN$'$ $=$ &c.; draw the ordinates and complete the parallelograms NC, N$'$P, &c.; then from the equation of the hyperbola we have BC $= \frac{1}{1^n}$, NP $= \frac{1}{2^n}$, N$'$P$'$ $= \frac{1}{3^n}$, &c., and the parallelogram NC, which $=$ BC $\times$ BN $= \frac{1}{1^n}$, N$'$P $= \frac{1}{2^n}$, &c. $=$ &c.; hence, the sum of the proposed series $=$ the sum of the parallelograms NC, N$'$P, &c., ad inf. But the sum of the parallelograms ad inf. is finite or infinite according as the hyperbolick area BVVC is finite or infinite, for their difference is finite, being less than NC, and consequently the series is finite only when n is greater than unity.

Cor. 2. Join AP, AC; then it may be shown that the hyperbolick sector ACP $=$ the area BCPN.

Ex. 6. Required the area of the hyperbolick sector OAP, the equation of the axis being given.



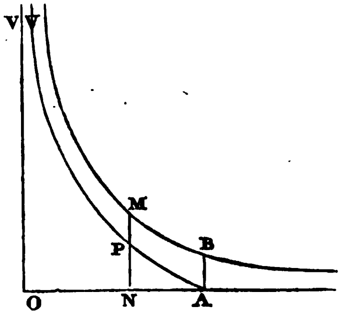
ON $= x$, $\therefore$ NP $= \frac{b}{a}\sqrt{x^2 - a^2}$

and $d.$ONP $= \dfrac{ydx + xdy}{2}$, but $d.$APN $= ydx$, $\therefore$ $d.$OAP $=$

$$\frac{xdy - ydx}{2} = \frac{b}{2a}\left\{\frac{x^2dx}{\sqrt{x^2 - a^2}} - \sqrt{x^2 - a^2}\,dx\right\} = \frac{ab}{2} \cdot \frac{dx}{\sqrt{x^2 - a^2}}$$

$$\left.\begin{array}{l}\therefore\ u = \dfrac{ab}{2}\,l(x + \sqrt{x^2 - a^2}) + \text{C} \\ \&\ \ o = \dfrac{ab}{2}\,l\,a + \text{C}\end{array}\right\} \therefore u = \frac{ab}{2}\,l.\,\frac{x + \sqrt{x^2 - a^2}}{a}.$$

Ex. 7. NP is always taken in the asymptote of an hyperbola such that AO × NP = the hyperbolick area ABMN; required the area traced by P.



ON $= x$ NP $= y$ OA $= b$	then AO × NP = ABMN $= a^2.\, l\dfrac{\text{OA}}{\text{ON}}$,

or $by = a^2.\, l\dfrac{a}{x}$, $\therefore\ bdy = -\dfrac{a^2dx}{x}$, $\therefore\ xdy = -\dfrac{a^2}{b}dx$,

$\therefore$ ANP, which $= -\int ydx = \int xdy - xy$ (2. 52.) $= \dfrac{-a^2x}{b} -$

$xy + c$, and $o = -a^2 + c$, $\therefore$ APN $= a^2 - \dfrac{x}{b}\ (a^2 + by)$,

$\therefore$ (5. 15. Ex. 10) AVVO $= a^2$, which is independent of OA.

Ex. 8. $y = al\dfrac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}}$, required the area between the values $x = a$ and $x = b$.

$$y = a.\, l.\frac{(a - \sqrt{a^2 - x^2})^2}{x^2} = 2a\left\{l.(a - \sqrt{a^2 - x^2} - lx\right\}$$

$$\therefore dy = 2a\left\{\frac{xdx}{\sqrt{a^2 - x^2}\,(a - \sqrt{a^2 - x^2})} - \frac{dx}{x}\right\} \ \ldots\ldots\ldots$$

$$= \frac{2a\,(a^2 - a\sqrt{a^2 - x^2})dx}{x\sqrt{a^2 - x^2}(a - \sqrt{a^2 - x^2})} = \frac{2a^2dx}{x\sqrt{a^2 - x^2}}\ \therefore\ u = \int ydx =$$

$$xy - \int xdy = xy - \int\frac{2a^2dx}{\sqrt{a^2 - x^2}} = xy - 2a^2 \times \sin.^{-1}\frac{x}{a}$$

$$+ c;\ \text{and}\ 0 = -2a^2 \times \frac{\pi}{2},\ \therefore\ u = xy + 2a^2 \times \cos.^{-1}\frac{x}{a}$$

$$\therefore\ (u) = 2ab\ l.\frac{a - \sqrt{a^2 - b^2}}{b} + 2a^2 \times \cos.^{-1}\frac{b}{a}.$$

Ex. 9. $x^3 - axy + y^3 = 0$.

Where the variables cannot be easily separated, as in this example, recourse must be had to substitution.

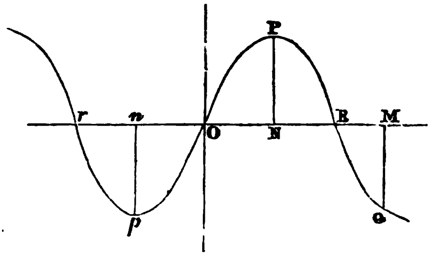
Substitute $y = \frac{ax^2}{z^2}$, $\therefore x^3 - \frac{a^2x^3}{z^2} + \frac{a^3x^6}{z^6} = 0$, or $1 - \frac{a^2}{z^2} + \frac{a^3x^3}{z^6} = 0$, $\therefore a^3x^3 = a^2z^4 - z^6$, $\therefore 3a^3x^2dx = (4a^2z^3 - 6z^5)dz$, $\therefore \int ydx = \int \left\{ \frac{4z}{3} - \frac{2z^3}{a^2} \right\} dz = \frac{2}{3}z^2 - \frac{1}{2}\frac{z^4}{a^2} = \frac{2}{3}\frac{ax^2}{y} - \frac{1}{2}\frac{x^4}{y^2} + \text{c}$.

Ex. 10. $a^5x^2y^2 - x^9 = a^6y^3$.

Substitute $y = \frac{x^2}{z}$ and area $= \frac{-a^6y^4}{4x^8} + \frac{2a^5y^3}{9x^6}$.

Ex. 11. $y = x - x^3$.

$\therefore ydx = xdx - x^3dx$ and $u = \frac{x^2}{2} - \frac{x^4}{4} + \text{c}$.



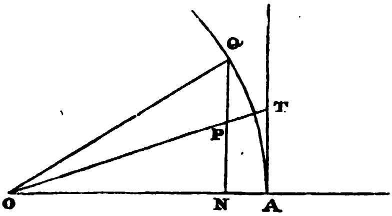
Take ON $= \frac{1}{\sqrt{3}} = .57\ldots$, and reckoning the area from the ordinate NP, we have
$\text{o} = \frac{1}{6} - \frac{1}{36} + \text{c}$, $\therefore \text{c} = -\frac{5}{36}$, $\therefore u = \frac{x^2}{2} - \frac{x^4}{4} - \frac{5}{36}$.

At a distance OR $= 1$, the curve cuts the axis; take OM $= \sqrt{\frac{5}{3}}$, then the area included between the ordinates NP and MQ $= \frac{5}{6} - \frac{25}{36} - \frac{5}{36} = 0$, which shows that the positive area PNR = the negative area RMQ. (Art. 1. Cor. 4.)

The area *opr* is similar and equal to OPR which $= \frac{1}{4}$; and since they are both positive, their sum $= \frac{1}{2}$, which is the result that would be obtained, were the value of u calculated between $x = +1$ and $x = -1$.

The origin in this example is the *centre* of the curve (7. 6. Def. 2.)

Ex. 12. Required the area of a curve traced by the intersection of the sine of an arc and the secant of half the arc.



Let the secant OT cut the sine NQ in P; join OQ; then (Eu. 6. 3.) NP : PQ :: NO : OQ ∴ comp^o. NP : NQ :: NQ : NO + OQ; or, by substitution, $y : \sqrt{a^2 - x^2} :: x : a+x$, $\therefore y = x\sqrt{\dfrac{a-x}{a+x}}$, $\therefore du = \dfrac{axdx}{\sqrt{a^2-x^2}} - \dfrac{x^2dx}{\sqrt{a^2-x^2}}$,

$$\therefore u = -a\sqrt{a^2-x^2} + \frac{1}{2}x\sqrt{a^2-x^2} - \frac{a^2}{2}\sin.^{-1}\frac{x}{a} + C$$

(2. 54. Ex. 2.). And correcting from the origin, $u = a^2 - \dfrac{2a-x}{2}\sqrt{a^2-x^2} - \dfrac{a^2}{2}\sin.^{-1}\dfrac{x}{a}$; hence the area of the *foliate*, as it is sometimes called, $= 2a^2 - a^2 . \dfrac{\pi}{2} = a^2\left(2 - \dfrac{\pi}{2}\right)$.

The area in the 2d and 3d quadrants $= a^2\left(2 + \dfrac{\pi}{2}\right)$ and the whole area of the curve $= 4a^2$.

Ex. 13. $y^2 + x^2 = \dfrac{a^2x}{2a-x}$.

$$y^2 = \frac{a^2x^2}{2ax-x^2} - x^2 = \frac{x^2(a-x)^2}{2ax-x^2}, \therefore y = \frac{x(a-x)}{\sqrt{2ax-x^2}}, \therefore$$

$du = \dfrac{x(a-x)dx}{\sqrt{2ax-x^2}}$, ∴ (2. 52.) $u = x\sqrt{2ax-x^2} - \ldots\ldots$ $\int dx\sqrt{2ax-x^2}$.

Suppose $x = a$, then half the area of the foliate $= a^2\left(1 - \dfrac{\pi}{4}\right)$; suppose $x = 2a$, then $u = -\dfrac{\pi a^2}{2}$, and consequently the area in the 4th quadrant terminated by the asymptote $= a^2\left(1 + \dfrac{\pi}{4}\right)$, and the whole area of the curve $= 4a^2$.

Ex. 14. $y = (a+x)\dfrac{\sqrt{2a-x}}{\sqrt{2a+x}}$ (Vid. Ch. 7. 14. Ex. 3.)

$$y = \frac{(a+x)(2a-x)}{\sqrt{4a^2-x^2}}, \therefore du = ydx = \frac{2a^2dx}{\sqrt{4a^2-x^2}} + \frac{axdx}{\sqrt{4a^2-x^2}}$$

$$-\frac{x^2dx}{\sqrt{4a^2-x^2}}, \therefore u = 2a^2 \sin.^{-1}\frac{x}{2a} - a\sqrt{4a^2-x^2} \quad \ldots\ldots$$

$$+\frac{1}{2}x\sqrt{4a^2-x^2} - 2a^2 \sin.^{-1}\frac{x}{2a} + \text{c (2. 54. Ex. 2.)} \ldots$$

$$= -\frac{2a - x}{2}\sqrt{4a^2-x^2} + \text{c, and, correcting from the origin,}$$

$$u = 2a^2 - \frac{2a-x}{2}\sqrt{4a^2-x^2}, \therefore (u) = 2a^2, \therefore \text{ the whole}$$

area in the 1st and 4th quadrants $= 4a^2$.

If we calculate the fluent between the values $x = 2a$ and $x = -2a$, which are the limits of the curve $(u) = 0$, which shows that the area in the 1st and 2d quadrants are equal, and consequently the whole area $= 8a^2$.

Ex. 15. *A circle.*

Let the origin be at the centre and the radius $= 1$.

$$y = \sqrt{1-x^2}, \therefore \text{OBPN} = \int (1-x^2)^{\frac{1}{2}} du = \int dx \left(1 - \frac{x^2}{2}\right.$$

$$\left. - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128} - \text{\&c.}\right) \therefore \text{OBPN} = x - \frac{x^3}{6} - \frac{x^5}{40} \ldots$$

$$-\frac{x^7}{112} - \frac{5x^9}{1152} - \text{\&c. for c} = \text{o}.$$

To compute this, let BP $= 30^\circ$, then $x = \frac{1}{2}$ and we have

$$\frac{x^3}{6} = .0208333 \ldots$$

$$\frac{x^5}{40} = .0007812 \ldots$$

$$\frac{x^7}{112} = .0000698 \ldots$$

$$\frac{5x^9}{1152} = .0000085 \ldots$$

$\frac{7x^{10}}{2560} = .0000012$

&c. = &c.

sum $= .0216940$, $\therefore$ OBPN $= .5 - .021694 = .478306$

Also, PN $= \sqrt{\frac{3}{4}}$, $\therefore \Delta$ONP $= \frac{\sqrt{3}}{8} \qquad = .216506$

$\therefore$ sector BOP $\qquad = .261800$

and the whole circle $= 12 \times$ sector BOP $= 3.1416\ldots$ which is accurate to 3 places of decimals. The area to 5 places is $3.14159\ldots$ for which π is generally substituted.

If the area had been calculated to a radius $= a$, then it would have proved $= \pi a^2$; which shows that circles vary in the duplicate ratio of their radii.

Cor. Since the corresponding ordinates of an ellipse, and of a circle described upon its axis major, are in the constant ratio of the axis minor to the axis major, their areas are in the same ratio. Hence, if a and b are the semi-axes major and minor of an ellipse, its area $= \frac{b}{a}.\ \pi a^2 = \pi.ab =$ the area of a circle whose radius is a mean proportional between the semi-axes.

Ex. 16. *Circular area forms.*

By means of the series which OBPN equals, we can approximate to its value in terms of the cosine or versed sine of the arc AP.

These approximate values have been calculated, and are registered in a table called a Table of Segments, which enables us to compute the value of any fluent of a *circular area* form. The circular area forms are as follows:

(1.) $\int \sqrt{a^2 - x^2}\, dx$ . . . .
$=$ cir. area, rad. $= a$, absc. $= x$.

$= a^2$. cir. area, rad. $= 1$, absc. $= \frac{x}{a}$.

(2.) $\int - \sqrt{a^2 - x^2}\, dx = a^2$. cir. area, cos. $= \frac{x}{a}$.

(3.) $\int \sqrt{2ax - x^2}\, dx = a^2$. cir. area, v.s. $= \frac{x}{a}$.

the ordinate of the curve is equal to the difference of the chord of the circular arc and of its versed sine; $\therefore (u) = \frac{a^2}{6}$.

14. A curve and a semicircle have the same abscissa, and the ordinate of the curve is equal to the tangent of half the circular arc; show that the area of the curve is equal to twice the corresponding circular segment.

15. A curve is traced by taking its ordinate = the arc of a circle and its abscissa = the sine; $\therefore (u) = a^2\left(\frac{\pi}{2} - 1\right)$.

16. A radius CNM is drawn cutting a chord of the circle in N; NP is always drawn at right angles to the chord and equal to NM; required the area traced by P.

$$u = cr + \frac{\sqrt{r^2 - c^2}}{2} l. \frac{r+c}{r-c}.$$

17. Required the area of a curve traced by a point which is taken in the secant of an arc at a distance from its extremity equal to the tangent. $(u) = a^2\left(1 - \frac{\pi}{4}\right)$.

This curve intersects the axis at $\angle\ 45^\circ$.

18. $ay^3 = x^2(by - x^2)\ \therefore u = \frac{b^2y^3}{3x^3} - \frac{4aby^5}{5x^5} + \frac{3a^2y^7}{7x^7}$.

This curve intersects the origin at $\tan.^{-1}\sqrt{\frac{b}{a}}$.

5. The preceding formula frequently fails in its application when the area includes an infinite ordinate.

Thus, let $y = \frac{a^{n+1}}{(a+x)^n}$ which is the equation of an hyberbola between the asymptotes; then, integrating from BC where AB $= a$ (Fig. 3. Ex. 5), we have . . . .

$$u = \frac{a^2}{n-1} \frac{(a+x)^{n-1} - a^{n-1}}{(a+x)^{n-1}}.$$

Suppose x to be a greater negative magnitude than $-a$, viz. $-(a+v)$, and that n is *even* and greater than unity; then the area consists of two infinite areas which have the same sign, for they are in the same quadrant, whose centre is B; and yet u, which $= \frac{a^2}{n-1} \cdot \frac{a^{n-1} + v^{n-1}}{v^{n-1}}$, is finite.

If n is *odd*, the analytical value of u does not fail; for, y changing its sign in its passage through infinity, the whole area is the difference of two infinite areas terminated by BC and by a line bc in the third quadrant, and is therefore properly expressed by $u = \frac{a^2}{n-1} \frac{v^{n-1} - a^{n-1}}{v^{n-1}}$.

From this instance we may conclude that, in general, in integrating $\int \mathrm{x}dx$ between $x = a$, $x = b$, if there is an intermediate value of the function which is infinite, the analytical expression does not necessarily represent the fluent.

6. It is sometimes required to compute an area, not from the equation of its curve, but from the value of a certain number of ordinates which are at given intervals from each other. In this case we can only approximate to the value of the area. By the method of interpolations, we can approximate to the intermediate ordinates till the intervals are so small that the included areas may be considered to be parabolick, which may therefore be calculated. For an explanation of this method, vid. Vince's Astronomy, Vol. 2, Art. 1248. Principia, Lib. 3, Lem. 5.

T. Simpson has given in his fluxions a very useful formula for calculating the parabolick areas.

7. *Required the length of the arc intercepted between two known co-ordinates of a given curve.*

AN $= x$ NP $= y$ AP $= s$	It has been shown (8. 2. Cors. 2 and 3) that if the co-ordinates are rectangular, $ds = \sqrt{dx^2 + dy^2}$, or $\frac{ds}{dx} = \sqrt{1+p^2}$, and that if they are inclined at

$\angle a$, $\frac{ds}{dx} = (p^2 - 2\cos.a\, p + 1)^{\frac{1}{2}}$.

Hence, from the given equation calculate the value of ds in terms either of x or of y; and the fluent, corrected if necessary, will be a right line which is equal to the arc AP.

8. *The rectification of curves by means of Taylor's Theorem.*

In fig. 8. 2. the arc Pp is always greater than the chord Pp and less than the tangent Ps; but from Taylor's Theorem, $p\pi = \frac{dy}{dx}\frac{h}{1} + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \&c. =$, by substitution, $ph + \mathrm{P}.h^2$ where P is a series containing the ascending powers of h;

whence (Eu. 1. 47.) $Pp^2 = h^2 + h^2 (p + Ph)^2 = h^2(1+p^2) + 2Pph^3 +$ &c. Also, since Ps is a tangent, $s\pi = p.h$; or $Ps^2 = (1+p^2)h^2$; whence we have

$$Ps^2 = (1+p^2)h^2$$

$$\text{arc}^2\ Pp = \left(\frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \text{\&c.}\right)^2 =, \text{ by substitution,}$$

$$\frac{du^2}{dx^2}h^2 + Qh^3$$

$$\text{chord}^2\ Pp = (1+p^2)h^2 + 2Pph^3 + \text{\&c.}$$

which series are in the order of their magnitudes whatever be the value of h, and consequently (3. 3.) $\frac{du^2}{dx^2} = 1 + p^2$ and $u = \int dx\sqrt{1+p^2}$.

Ex. 1. *A semi-cycloid.*

The equation is $y = z + \sqrt{2ax - x^2}$ where z is a circular arc v.s. $= x$ and $2a =$ the axis of the cycloid; differentiating,

$$p = \frac{a}{\sqrt{2ax - x^2}} + \frac{a-x}{\sqrt{2ax-x^2}} = \sqrt{\frac{2a-x}{x}}, \therefore s = \int dx\sqrt{1+p^2}$$

$= \int dx\sqrt{\frac{2a}{x}} = 2(2ax)^{\frac{1}{2}} =$ twice the corresponding chord of the circle.

Ex. 2. *A semi-cubical parabola.*

The equation is $x = \frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$. Here it is more convenient to calculate ds in terms of y; hence, we have . . . .

$$dx = \frac{3}{2}\frac{y^{\frac{1}{2}}dy}{a^{\frac{1}{2}}} \text{ and } ds^2 = \frac{9ydy^2}{4a} + dy^2 \therefore ds = dy.\frac{(9y+4a)^{\frac{1}{2}}}{2a^{\frac{1}{2}}}$$

$$\left.\begin{aligned} \therefore s &= \frac{(9y+4a)^{\frac{3}{2}}}{27a^{\frac{1}{2}}} + C \\ o &= \frac{(4a)^{\frac{3}{2}}}{27a^{\frac{1}{2}}} + C \end{aligned}\right\} \therefore s = \frac{(9y+4a)^{\frac{3}{2}}}{27a^{\frac{1}{2}}} - \frac{8a}{27}.$$

To obtain s in terms of x, $a^{\frac{1}{3}}\,x^{\frac{2}{3}}$ must be substituted for y in the above expression.

Ex. 3. The *Apollonian parabola.*

Here $ax = y^2$, $\therefore dx = \dfrac{2ydy}{a}$, $\therefore ds^2 = \dfrac{dy^2(4y^2 + a^2)}{a^2}$,

$\therefore ds = \dfrac{\sqrt{4y^2 + a^2}.\,dy}{a}$.

To integrate this, substitute $b = \dfrac{a}{2}$, $\therefore ds = \dfrac{(y^2 + b^2)^{\frac{1}{2}}dy}{b}$

$\therefore$ (2. 55. Ex. 1.) $s = \dfrac{y}{2b}\sqrt{y^2 + b^2} + \dfrac{b}{2}\,l\,\dfrac{y + \sqrt{y^2 + b^2}}{b}$,

and by means of a table of logarithms the length may be computed corresponding to any value of the co-ordinates.

Ex. 4. A parabola whose equation is $a^{n-1}y = x^n$.

$dy = \dfrac{nx^{n-1}dx}{a^{n-1}}$, $\therefore ds = dx\sqrt{1 + \dfrac{n^2x^{2n-2}}{a^{2n-2}}}$, $\therefore$

$$s = x + \frac{n^2x^{2n-1}}{(2n-1)\,2a^{2n-2}} - \frac{n^4x^{4n-3}}{(4n-3)8a^{4n-4}} + \frac{n^6x^{6n-5}}{(6n-5)\,16a^{6n-6}}$$
$- \&c.$

Comparing this fluent with the form Ch. 2. Art. 46. we have $\left.\begin{array}{l} n = 2n-2 \\ rn - 1 = 0 \end{array}\right\} \therefore r = \dfrac{1}{2n-2}$, and consequently the series will terminate when $\dfrac{1}{2n-2}$ is a positive integer; and to find the requisite values of n, let $v = \dfrac{1}{2n-2}$, $\therefore n = \dfrac{1+2v}{2v}$, in which substitute the numbers 1, 2, 3, &c. and we have $n = \frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \frac{9}{8}, \frac{11}{10}$, &c.; in all which cases the length of the parabola can be expressed in a finite algebraical form.

The series likewise terminates when $m + r$ or $\dfrac{1}{2} + \dfrac{1}{2n-2}$ is a negative integer; let $-v = \dfrac{1}{2} + \dfrac{1}{2n-2} = \dfrac{n}{2n-2}$

$\therefore n = \dfrac{2v}{1+2v}$; or the parabola possesses the same property when $n = \frac{2}{3}, \frac{4}{5}, \frac{6}{7}$, &c.

It is obvious that the latter series give the same parabolas as the former, but with their axes changed.

Ex. 5. *A circle.*

Let $t = \tan. s$, then (1. 32. form 4.) $ds = \dfrac{dt}{1+t^2} = \ldots\ldots$ $dt\,(1 - t^2 + t^4 - \&c.)$ $\therefore\ s = t - \dfrac{t^3}{3} + \dfrac{t^5}{5} - \dfrac{t^7}{7} + \&c.$ for $c = 0$.

And to compute this, suppose $s = 30^\circ = \frac{1}{12}$th of the whole circumference; then $t = \dfrac{1}{\sqrt{3}} = .5773502\ldots.$ Substitute this in the series, and taking 12 terms, $s = .5235987$ .... therefore the whole circumference $= 6.2831804\ldots. = 2\pi$.

If the circumference is computed to radius $= r$, it $= 2\pi r$; or circumferences of circles vary as their radii.

Since we obtain only an approximate value of s, the circle is not a rectifiable curve.

To obtain a series which converges with greater rapidity, the following artifice may be adopted.

$$\left.\begin{aligned}\text{Let } A &= \tan.^{-1}\tfrac{1}{2}\\ B &= \tan.^{-1}\tfrac{1}{3}\end{aligned}\right\}\therefore\ \tan.A + \tan.B = \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{5}{6},$$

and $\tan.(A+B) = \dfrac{\frac{5}{6}}{1-\frac{1}{6}} = 1$, $\therefore\ A+B = 45^\circ$;

Hence, since $s = t - \dfrac{t^3}{3} + \dfrac{t^5}{5} - \&c.$

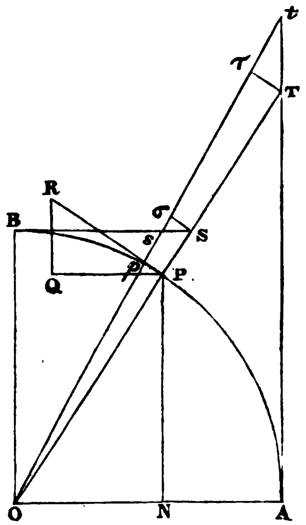
$$A = \frac{1}{2} - \frac{1}{3.2^3} + \frac{1}{5.2^5} - \&c.$$

$$B = \frac{1}{3} - \frac{1}{3.3^3} + \frac{1}{5.3^5} - \&c.$$

$$\therefore\ 45^\circ = A + B = \frac{5}{6} - \left(\frac{1}{3.2^3} + \frac{1}{3.3^3}\right) + \left(\frac{1}{5.2^5} + \frac{1}{5.3^5}\right) - \&c.,$$

which is $\frac{1}{8}$th of the whole circumference.

Ex. 6. Required to deduce the formulæ of Ch. 1. 32. upon geometrical principles.



Let AB be a quadrant whose centre is O; take any arc AP and draw its trigonometrical lines as in the figure; draw the secant opt near to OPT, and Tτ, sσ perpendicular on Ot. Also draw the fluxional triangle PQR.

OA $= a$ ON $= x$ NP $= y$ AT $= t$ OT $= s$ AP $= u$	(1.) From $\triangle$ PQR, OPN, PR $= \dfrac{\text{RQ} \times \text{PO}}{\text{ON}}$; or $du = \dfrac{ady}{\sqrt{a^2 - y^2}}$

which is form (1).

(2.) Since ON decreases as AP increases, PQ in this case $= -dx$, and PR $= \dfrac{\text{PQ} \times \text{PO}}{\text{PN}}$; or $du = \dfrac{-adx}{\sqrt{a^2-x^2}}$.

(3.) The fluxion of the versed sine may be deduced from that of the cosine.

(4.) $\left.\begin{array}{r} \text{P}p : \text{T}\tau :: \text{OP} : \text{OT} \\ \text{and in the limit, } \text{T}\tau : \text{T}t :: \text{OA} : \text{OT} \end{array}\right\} \therefore$ in the limit, $\text{P}p : \text{T}t ::$ $\text{OA}^2 : \text{OT}^2$; or $du : dt :: a^2 : a^2 + t^2$, $\therefore du = \dfrac{a^2dt}{a^2 + t^2}$.

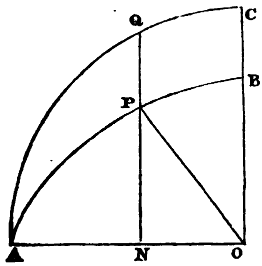
(5.) Let BS $= t$ then

$\left.\begin{array}{r} \text{P}p : \text{s}\sigma :: \text{OP} : \text{OS} \\ \text{and in the limit, } \text{s}\sigma : \text{sS} :: \text{OB} : \text{OS} \end{array}\right\} \therefore$ in the limit, $\text{P}p : \text{sS} ::$ $\text{OA}^2 : \text{OS}^2$; or $dz : -dt :: a^2 : a^2+t^2$, $\therefore dz = -\dfrac{a^2dt}{a^2 + t^2}$.

(6.) $\left.\begin{array}{r} \text{P}p : \text{T}\tau :: \text{OP} : \text{OT} \\ \text{and in the limit, } \text{T}\tau : t\tau :: \text{OA} : \text{AT} \end{array}\right\} \therefore$ in the limit, $\text{P}p : t\tau ::$ $\text{OA}^2 : \text{OT} \times \text{AT}$; or $du : ds :: a^2 : s\sqrt{s^2-a^2}$, $\therefore du = \dfrac{a^2ds}{s\sqrt{s^2-a^2}}$.

(7.) Let OS $= s$ then

$\left.\begin{array}{r} \text{P}p : \text{s}\sigma :: \text{OP} : \text{OS} \\ \text{and in the limit, } \text{s}\sigma : s\sigma :: \text{OB} : \text{SB} \end{array}\right\} \therefore du : -ds, :: a^2 : \ldots$ $s\sqrt{s^2 - a^2}$, or $du = -\dfrac{a^2ds}{s\sqrt{s^2 - a^2}}$.



Ex. 7. An *ellipse.*

AO the principal semi-axis $=a=1$
OB the transverse semi-axis $=b$
The eccentricity $=e$
ON $=x$, NP $=y$, BP $=s$, OP $=r$.

The equation is $y=\sqrt{1-e^2}\sqrt{1-x^2}$ (7. 14. Ex. 2.)

$\therefore dy=-\sqrt{1-e^2}\,\dfrac{xdx}{\sqrt{1-x^2}}$, $\therefore ds=dx\sqrt{1+\dfrac{(1-e^2)x^2}{1-x^2}}$

$=dx\sqrt{\dfrac{1-e^2x^2}{1-x^2}}$.

But $\sqrt{1-e^2x^2}=1-\dfrac{1}{2}e^2x^2-\dfrac{1.1}{2.4}e^4x^4-\dfrac{1.1.3}{2.4.6}e^6x^6-$&c.
a series which converges rapidly if e is a small fraction; whence, we have $s=\int\dfrac{dx}{\sqrt{1-x^2}}\left\{1-\dfrac{1}{2}e^2x^2-\dfrac{1.1}{2.4}e^4x^4-\dfrac{1.1.3}{2.4.6}e^6x^6-\&c.\right\}$; each of which can be integrated either by Hirsch's Tab. 25. Irrat. Diff., or from a formula which will be demonstrated in the 2d vol. viz.

$$\int\frac{x^m dx}{\sqrt{1-x^2}}=-\frac{(1-x^2)^{\frac{1}{2}}x^{m-1}}{m}+\frac{m-1}{m}\int\frac{x^{m-2}dx}{\sqrt{1-x^2}},$$ which at each integration reduces the index of x without the vinculum by two; thus we have

$$\int\frac{x^2dx}{\sqrt{1-x^2}}=-\tfrac{1}{2}x\sqrt{1-x^2}+\tfrac{1}{2}\int\frac{dx}{\sqrt{1-x^2}}$$

$$=-\tfrac{1}{2}x\sqrt{1-x^2}+\tfrac{1}{2}\text{A, if A}=\sin.^{-1}x$$

$$\int\frac{x^4dx}{\sqrt{1-x^2}}=-\tfrac{1}{4}x^3\sqrt{1-x^2}+\tfrac{3}{4}\int\frac{x^2dx}{\sqrt{1-x^2}}\ \ldots\ldots\ldots$$

$$=-\left(\frac{x^2}{4}+\frac{1.3}{2.4}\right)x\sqrt{1-x^2}+\frac{1.3}{2.4}\text{A}.$$

$$\int\frac{x^6dx}{\sqrt{1-x^2}}=-\tfrac{1}{6}x^5\sqrt{1-x^2}+\tfrac{5}{6}\int\frac{x^4dx}{\sqrt{1-x^2}}\ \ldots\ldots$$

$$= -\left(\frac{x^4}{4} + \frac{1.5}{4.6}x^2 + \frac{1.3.5}{2.4.6}\right)x\sqrt{1-x^2} + \frac{1.3.5}{2.4.6}\text{A}.$$

&c. = &c.;

and substituting these in the series which expresses the value of s we have

$$s = \text{A} + \frac{1}{2}e^2\left\{\frac{1}{2}\sqrt{1-x^2} - \frac{1}{2}\text{A}\right\} + \frac{1.1}{2.4}e^4\left\{\frac{x^2}{4} + \frac{1.3}{2.4}x\sqrt{1-x^2} - \frac{1.3}{2.4}\text{A}\right\} + \frac{1.1.3}{2.4.6}e^6\left\{\left(\frac{x^4}{6} + \frac{1.5}{4.6}x^2 + \frac{1.3.5}{2.4.6}\right)x\sqrt{1-x^2} - \frac{1.3.5}{2.4.6}\text{A}\right\} + \text{\&c.}$$

Cor. 1. If the principal semi-axis $= a$, $ds = dx\sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}}$.

Cor. 2. When b is greater than a, the eccentricity of the ellipse is measured on the transverse axis and ds, which $= dx\sqrt{1 + \frac{b^2}{a^2}\,\frac{x^2}{a^2 - x^2}}, = \frac{dx}{a}\sqrt{\frac{a^4 + (b^2 - a^2)x^2}{a^2 - x^2}} = dx\sqrt{\frac{1 + e^2b^2x^2}{1 - x^2}}$ when $a = 1$.

Cor. 3. If s is the arc of an hyperbola beginning at the vertex, it may be shewn as in the example that $ds = \frac{dx\sqrt{e^2x^2 - 1}}{\sqrt{x^2 - 1}}$ where $e^2 = 1 + b^2$.

Cor. 4. $ds = \frac{dy}{b}\sqrt{\frac{b^4 + e^2y^2}{b^2 \mp y^2}}$, the lower sign belonging to the hyperbola.

For the equations are $x^2 \pm \frac{y^2}{b^2} = 1$, $\therefore x = \frac{\sqrt{b^2 \mp y^2}}{b}$ and

$$dx = \frac{\mp y\,dy}{b\sqrt{b^2 \mp y^2}} \text{ and } ds = -dy\sqrt{1 + \frac{y^2}{b^2(b^2 \mp y^2)}} \quad \ldots\ldots$$

$$= -\frac{dy}{b}\sqrt{\frac{b^4 + e^2y^2}{b^2 \mp y^2}} = -\frac{dy}{b}\sqrt{\frac{b^4 + e^2a^2y^2}{b^2 \mp y^2}}.$$

Cor. 5. If the origin is placed at the vertex of the ellipse, or of the hyperbola, and s is the arc beginning at the vertex, $ds = dx\sqrt{\frac{b^2 + e^2(2ax \mp x^2)}{2ax \mp x^2}}$, the lower sign belonging to the hyperbola.

Cor. 6. $ds = \frac{rdr\sqrt{a^2+b^2-r^2}}{\sqrt{(r^2-b^2)(a^2-r^2)}}$ where r is the radius vector of the ellipse.

For $r^2 = x^2 + y^2 = x^2 + b^2(1-x^2) = b^2 + e^2x^2 \ldots.$

$\therefore x = \frac{\sqrt{r^2-b^2}}{e}$, $\therefore ds$, which $= dx\sqrt{\frac{1-e^2x^2}{1-x^2}} \ldots\ldots$

$$= \frac{rdr}{e\sqrt{r^2-b^2}}\sqrt{\frac{1-(r^2-b^2)}{1-\frac{r^2-b^2}{e^2}}} = \frac{rdr\sqrt{1+b^2-r^2}}{\sqrt{(r^2-b^2)(e^2+b^2-r^2)}}$$

$$= \frac{rdr\sqrt{a^2+b^2-r^2}}{\sqrt{(r^2-b^2)(a^2-r^2)}}.$$

Cor. 7. In the hyperbola $ds = \frac{rdr\sqrt{r^2-a^2+b^2}}{\sqrt{(r^2+b^2)(r^2-a^2)}}$.

Cor. 8. $ds = d\phi\sqrt{1-e^2\sin.^2\phi}$, where $\phi =$ the corresponding circular arc described upon the axis major of the ellipse as a diameter.

For produce OB, NP to meet the circle in C and Q, let $CQ = \phi$; then $x = \sin.\phi \therefore d\phi = \frac{dx}{\sqrt{1-x^2}}$ or $dx = \sqrt{1-x^2}.d\phi$;

whence ds, which $= \frac{dx\sqrt{1-e^2x^2}}{\sqrt{1-x^2}}$, $= d\phi\sqrt{1-e^2\sin.^2\phi}$.

This form may also be integrated by expanding it in a series by Taylor's Theorem.

Since $\sin.^2\phi = \frac{1}{2} - \frac{1}{2}\cos.2\phi$ (Trig. p. 62), the fluent may be put under the form $dz(1-n\cos.z)^{\frac{1}{2}}$ where $z = 2\phi$ and $n = \frac{e^2}{2-e^2}$, which is more convenient for the developement.

The form $\frac{d\phi}{\sqrt{1-e^2\sin.^2\phi}}$ may be integrated in the same manner by expanding it in a series.

9. *Required to compute the periphery of an ellipse.*

Since the fluxion of an elliptick arc contains a circular form, we can only obtain an approximate value. Series for this purpose have been investigated by several eminent mathematicians, which converge with greater or less rapidity according to the value of e. When e is very small, the series of the preceding article will be found useful. Let

P = the periphery, then making $x = 1$, $A = \frac{\pi}{2}$, we have

$$\frac{P}{4} = \frac{\pi}{2}\left(1 - \frac{1}{2}e^2 - \frac{1.1.1.3}{2.2.4.4}e^4 - \frac{1.1.1.3.3.5}{2.2.4.4.6.6}e^6 - \&c.\right)$$

The series given by Euler in the 2d vol. of his Opuscula is of great utility, since it converges the more rapidly the greater the eccentricity is. The circumference of an ellipse whose eccentricity is .99, computed by means of this series = 4.1139 .... *

10. *Elliptick and hyperbolick forms investigated.*

If these are integrated and their fluents computed and registered in tables, it is obvious that they may be applied to the same use in the integration of other fluents which we have seen may be derived from the circular or the logarithmick forms. We shall proceed therefore to investigate such other elementary forms as occur the most frequently in calculation.

(1.) In $ds = \frac{dx\sqrt{1-e^2x^2}}{\sqrt{1-x^2}}$, substitute $v = \sqrt{1-e^2x^2}$ . .

$\therefore x = \frac{\sqrt{1-v^2}}{e}$ $\therefore dx = \frac{-vdv}{e\sqrt{1-v^2}}$ and $\sqrt{1-x^2}$ . . . . . .

$= \sqrt{1 - \frac{1-v^2}{e^2}} = \frac{\sqrt{v^2-b^2}}{e}$; $\therefore$ we have . . . .

$$ds = \frac{-v^2dv}{\sqrt{(v^2-b^2)(1-v^2)}} = \frac{-v^2dv}{\sqrt{-v^4+(1+b^2)v^2-b^2}}.$$

In this formula the transverse axis is supposed to be the *minor* axis of the ellipse: if b is greater than a, $ds = \frac{dx\sqrt{1+e^2b^2x^2}}{\sqrt{1-x^2}}$, in which if $v = \sqrt{1+e^2b^2x^2}$ be substituted, there results $ds = \frac{v^2dv}{\sqrt{-v^4+(1+b^2)v^2-b^2}}$, the same formula as before, except that the sign of the numerator is changed.

Cor. Hence $\frac{-x^2dx}{\sqrt{Ax^2-x^4-B}}$, where A and B are positive quantities, may be integrated.

For comparing it with the above form, which when reduced to an axis major = $2a$ becomes

* See a paper in the Philosophical Transactions for 1804 by Professor Woodhouse.

$s = \int \frac{-v^2 dv}{\sqrt{-v^4 + (a^2+b^2)v^2 - a^2b^2}}$, we have $A = a^2 + b^2$, $B = a^2b^2$; whence $a + b = \sqrt{A + 2\sqrt{B}} + \sqrt{A - 2\sqrt{B}}$; and $a - b = \sqrt{A + 2\sqrt{B}} - \sqrt{A - 2\sqrt{B}}$, or $2a = \sqrt{A + 2\sqrt{B}}$ and $2b = \sqrt{A - 2\sqrt{B}}$; wherefore $\int \frac{x^2 dx}{\sqrt{Ax^4 - x^3 - B}} =$ elliptick arc, axes $= \sqrt{A + 2\sqrt{B}} \pm \sqrt{A - 2\sqrt{B}}$, and absc. $= \frac{\sqrt{1 - x^2}}{e} = \frac{\sqrt{1 - x^2}}{\sqrt{1 - b^2}} = \frac{a\sqrt{a^2 - x^2}}{\sqrt{a^2 - b^2}}$.

If a is less than b the fluent must be constructed by means of an ellipse whose transverse axis is the axis *major;* the magnitude of the axes will be the same as before, viz. $\sqrt{A + 2\sqrt{B}} \mp \sqrt{A - 2\sqrt{B}}$, but the abscissa of the arc now $= \frac{a\sqrt{x^2 - a^2}}{\sqrt{b^2 - a^2}}$.

(2.) In $ds = \frac{dx\sqrt{e^2x^2 - 1}}{\sqrt{x^2 - 1}}$ substitute $v = \sqrt{e^2x^2 - 1}$, ∴ $x = \frac{\sqrt{v^2 + 1}}{e}$ and $ds = \frac{vdv}{e\sqrt{v^2 + 1}} \times \frac{v}{\sqrt{\frac{v^2 + 1}{e^2} - 1}} \ldots\ldots$

$$= \frac{v^2 dv}{\sqrt{(v^2 - b^2)(v^2 + 1)}} = \frac{v^2 dv}{\sqrt{v^4 + (1 - b^2)v^2 - b^2}}.$$

Hence $\int \frac{x^2 dx}{\sqrt{x^4 + Ax^2 - B}} =$ hyp. arc, $\frac{1}{2}$ axes $\ldots\ldots$

$$= \sqrt{\pm \frac{A}{2} + \sqrt{\frac{A^2}{4} + B}},\ \text{absc.} = \frac{a\sqrt{a^2 + x^2}}{\sqrt{a^2 + b^2}}.$$

Cor. If the hyperbola is equilateral, $ds = \frac{v^2 dv}{\sqrt{v^4 - a^4}}$.

(3.) In $ds = \frac{dx\sqrt{1 \sim e^2x^2}}{\sqrt{1 \sim x^2}}$ substitute $\frac{b^2}{v^2} = 1 \sim e^2x^2$, and there results $ds = \frac{\pm b^2 dv}{v^2\sqrt{(v^2 \mp b^2)(1 - v^2)}} \ldots\ldots\ldots\ldots$

$$= \frac{\pm b^2 dv}{v^2\sqrt{-v^4 + (1 \pm b^2)v^2 \mp b^2}}.$$

If b is greater than 1, $ds = \frac{-b^2 dv}{v^2\sqrt{-v^4+(b^2+1)v^2-b^2}}$.

Cor. If the hyperbola is equilateral, $ds = \frac{-a^4 dv}{v^2\sqrt{a^4-v^4}}$.

(4). In $ds = \frac{dx\sqrt{1-e^2x^2}}{\sqrt{1-x^2}}$, substitute $1-vx = \sqrt{1-x^2}$

$\therefore x = \frac{2v}{1+v^2} \therefore dx = \frac{2dv(1-v^2)}{(1+v^2)^2}$; and $\sqrt{1-x^2}$ .....

$= 1 - \frac{2v^2}{1+v^2} = \frac{1-v^2}{1+v^2} \therefore ds = \frac{2dv}{1+v^2}\sqrt{1-\frac{4e^2v^2}{(1+v^2)^2}}$ ..

$= \frac{2dv}{(1+v^2)^2}\sqrt{v^4-2(2e^2-1)v^2+1}$.

Cor. 1. Hence $\int\frac{2dx}{(1+x^2)^2}\sqrt{x^4-\text{A}x^2+1}$ may be integrated; for, comparing co-efficients, $\frac{\text{A}}{2}+1 = 2e^2 = 2(1-b^2) \therefore 4b^2 = 2-\text{A}$ and $2b = \sqrt{2-\text{A}}$; wherefore $\int\frac{2dx}{(1+x^2)^2}\sqrt{x^4-\text{A}x^2+1}$ $=$ elliptick arc, axes $= 2$ and $\sqrt{2-\text{A}}$, absc. $= \frac{2x}{1+x^2}$.

Cor. 2. If the same substitution be made in $ds = \ldots$ $\frac{dx\sqrt{1+e^2b^2x^2}}{\sqrt{1-x^2}}$, there will result $\int\frac{2dx}{(1+x^2)^2}\sqrt{x^4+\text{A}x^2+1}$ $=$ ellipt. arc, axes $= 2$ and $\sqrt{2+\text{A}}$, absc. $= \frac{2x}{1+x^2}$.

11. *A table of elliptick and hyperbolick forms.*

(1.) $\int\frac{dx\sqrt{a^2\sim e^2x^2}}{\sqrt{a^2\sim x^2}} = a \times$ arc, $\frac{1}{2}$ axes $= 1$ and $\sqrt{1\sim e^2}$, absc. $= \frac{x}{a}$.

(2.) $\int\frac{dx\sqrt{1+m^2x^2}}{\sqrt{1-x^2}} =$ ellipt. arc, $\frac{1}{2}$ axes $= 1$ and $\sqrt{1+m^2}$, absc. $= x$.

(3.) $\int \frac{dx\sqrt{n^4+m^2x^2}}{\sqrt{n^2\mp x^2}} = n \times \text{arc}$, $\frac{1}{2}$ axes $= \sqrt{m^2 \pm n^2}$, and n, ord. $= x$.

(4.) $\int \frac{dx\sqrt{n^2+2mx\mp x^2}}{\sqrt{2mx\mp x^2}} = \sqrt{m^2\pm n^2} \times \text{arc}$, $\frac{1}{2}$ axes $=1$ and $\frac{n}{\sqrt{m^2\pm n^2}}$, absc. $= \frac{x}{m}$.

(5.) $\int \frac{xdx\sqrt{b^2+a^2\sim x^2}}{\sqrt{(x^2\mp b^2)\,(a^2\sim x^2)}} = \text{arc}$, $\frac{1}{2}$ axes $= a$ and b, abcs. $= \frac{a\sqrt{x^2\mp b^2}}{\sqrt{a^2\mp b^2}}$.

(6.) $\int dx\sqrt{1\mp m^2\sin.^2x} =$ ellipt. arc, $\frac{1}{2}$ axes $= 1$ and $\sqrt{1\mp m^2}$, absc. $= \sin.x$.

(7.) $\int \frac{-x^2dx}{\sqrt{-x^4+\text{A}x^2-\text{B}}} =$ ellipt. arc, axes $= \sqrt{\text{A}+2\sqrt{\text{B}}} \pm \sqrt{\text{A}-2\sqrt{\text{B}}}$, absc. $= \frac{a\sqrt{a^2-x^2}}{\sqrt{a^2\sim b^2}}$.

(8.) $\int \frac{x^2dx}{\sqrt{x^4+\text{A}x^2-\text{B}}} =$ hyp. arc, $\frac{1}{2}$ axes $= \ldots\ldots\ldots$ $\sqrt{\pm\frac{\text{A}}{2}+\sqrt{\frac{\text{A}^2}{4}+\text{B}}}$, absc. $= \frac{a\sqrt{a^2+x^2}}{\sqrt{a^2+b^2}}$.

(9.) $\int \frac{x^2dx}{\sqrt{x^4-a^4}} =$ equilat. hyp. arc, $\frac{1}{2}$ axis $= a$, absc. $= \sqrt{\frac{a^2+x^2}{2}}$.

(10.) $\int \frac{dx}{x^2\sqrt{-x^4+\text{A}x^2-\text{B}}} = \frac{1}{16}\text{B}$. ellipt. arc, $\frac{1}{2}$ axes $= \sqrt{\text{A}+2\sqrt{\text{B}}} \pm \sqrt{\text{A}-2\sqrt{\text{B}}}$, absc. $= \frac{a^2\sqrt{x^2\sim b^2}}{x\sqrt{a^2\sim b^2}}$.

(11.) $\int \frac{-dx}{x^2\sqrt{-x^4\pm\text{A}x^2+\text{B}}} = \frac{1}{\text{B}}$. hyp. arc, $\frac{1}{2}$ axes $= \sqrt{\pm\frac{\text{A}}{2}+\sqrt{\frac{\text{A}^2}{4}+\text{B}}}$, absc. $= \frac{a^2\sqrt{x^2+b^2}}{x\sqrt{a^2+b^2}}$.

(12.) $\int \frac{-a^4 dx}{x^2\sqrt{a^4 - x^4}}$ = hyp. arc, $\frac{1}{2}$ axis $= a$, absc. $=$ . . $\frac{a}{x}\sqrt{\frac{a^2+x^2}{2}}$.

(13.) $\int \frac{2dx}{(1+x^2)^2}\sqrt{x^4 \mp \text{A}x^2 + 1}$ = ellipt. arc, axes $= 2$ and $\sqrt{2 \mp \text{A}}$, absc. $= \frac{2x}{1+x^2}$.

In these forms the upper sign belongs to the ellipse and the lower to the hyperbola.

In forms (7.) and (10.) A and B must be both positive, and $\frac{\text{A}^2}{4}$ must be greater than B^2, or the roots of the equation $-x^4 + \text{A}x^2 - \text{B} = 0$ must be possible.

In forms (8.) and (11.) B must be positive, but A may be either positive or negative.

In form (13.) A may be either positive or negative, but it may not be a greater negative quantity than -2.

12. *Examples.*

Ex. 1. $du = \frac{dx\sqrt{b^2-x^2}}{\sqrt{a^2-x^2}}$, $\therefore u = \frac{b}{a}$ ellipt. arc, $\frac{1}{2}$ axes $= 1$ and $\frac{\sqrt{b^2-a^2}}{b}$, absc. $= \frac{x}{a}$.

Ex. 2. $du = \frac{x^{\frac{1}{2}}dx}{\sqrt{x^2+\text{A}x+\text{B}}}$.

Substitute $z^2 = x$, $\therefore du = \frac{2z^2dz}{\sqrt{z^4+\text{A}z^2+\text{B}}}$, $\therefore$ (10. form 8.)

$u = 2$ hyp. arc, $\frac{1}{2}$ axes $= \sqrt{\pm\frac{\text{A}}{2} + \sqrt{\frac{\text{A}^2}{4} - \text{B}}}$, absc. $=$ . . $\frac{a\sqrt{a^2+x}}{\sqrt{a^2+b^2}}$.

Ex. 3. $du = \frac{x^{\frac{1}{2}}dx}{\sqrt{x^2-1}}$.

By substitution $du = \frac{2z^2dz}{\sqrt{z^4-1}}$, $\therefore u = 2$ hyp. arc, $\frac{1}{2}$ axis $= 1$, absc. $= \sqrt{\frac{1+x}{2}}$.

Ex. 4. $du = \frac{x^{\frac{1}{2}}dx}{\sqrt{3x - x^2 - 2}}$.

By substitution $du = \frac{2z^2dz}{\sqrt{-z^4 + 3z^2 - 2}}$, $\therefore u = 2$ ellipt. arc, axes $= \sqrt{3 + 2\sqrt{2}} \pm \sqrt{3 - 2\sqrt{2}} =$ (Alg. 258.) $1 + \sqrt{2} \pm (1 - \sqrt{2}) = 2$ and $2\sqrt{2}$, absc. $= \sqrt{x-1}$.

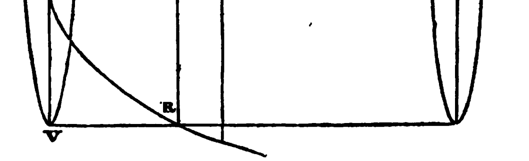
Ex. 5. $du = \frac{dx}{(1+x^2)}\sqrt{1+x^4}$.

$u = \frac{1}{2}$ ellipt. arc, axes $= 2$ and $\sqrt{2}$, absc. $= \frac{2x}{1+x^2}$.

Ex. 6. $du = dx\sqrt{\frac{1+x^6}{(1+x^2)^5}} = \frac{dx}{(1+x^2)^2}\sqrt{\frac{1+x^6}{1+x^2}} \ldots\ldots$ $= \frac{dx}{(1+x^2)^2}\sqrt{1-x^2+x^4}$.

$2u =$ ellipt. arc, $\frac{1}{2}$ axes $= 1$ and $.5$, absc. $= \frac{2x}{1+x^2}$.

13. *Required the content of a solid of revolution.*



All its sections perpendicular to the axis are circles.
Let PAR be the curve by whose revolution round the axis

AE the solid is formed. Construct the figure as in Art. 1., then the parallelograms NL, NQ, Np, Nπ generate cylinders.

Since the cylinder LN and the solid PAR may be supposed to be generated by circles moving parallel to themselves along the axis AE, we have inc. u : inc. cylinder LN :: solid generated by NPpn : cylinder Pn : but since cylinder mn : cylinder Pn :: circle Nm : circle NP :: Nm^2 : NP^2 (ultimately) :: 1 : 1; therefore, à fortiori, in the limit, the solid generated by NPpn : the cylinder Pn :: 1 : 1; whence, $du = d.$ cylinder LN = circle NP $\times$ $d.$ AN = cylinder NQ $= \pi y^2 dx$ and $u = \pi \int y^2 dx$.

Cor. Hence the fluxion of a solid is as the area of the generating circle and its velocity conjointly. Generally the fluxion of a solid generated by any plane perpendicular to an axis is as the area of the plane and its velocity.

14. *Required the content of a solid by Taylor's Theorem.*

Let P be the generating area which is supposed to increase or decrease by some certain law; and let u be the corresponding solid; then P and u may each be considered as functions of the abscissa x. When x becomes $x+h$, let P and u become P$'$ and U, then U $-$ u is always less than P$'h$ and greater than Ph; or $P'h = h\left\{P + \frac{dP}{dx}h + \&c.\right\}$, $\frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \&c.$, and P$h$ are in the order of their magnitudes whatever be the value of h; consequently (3.3) $\frac{du}{dx} = P$ and $u = \int P dx$.

The generating area is at right angles to the axis.

15. *Examples.*

Ex. 1. A paraboloid.

$$= ax \therefore du = \pi y^2 dx = \pi ax dx \therefore u = \frac{\pi a x^2}{2} = \frac{\pi y^2 x}{2};$$

i. e. the paraboloid $= \frac{1}{2}$ the circumscribing cylinder.

Ex. 2. Required the content of a paraboloid intercepted between two known abscissæ b and c.

$$\left.\begin{aligned} u &= \frac{\pi a x^2}{2} + C \\ o &= \frac{\pi a b^2}{2} + C \end{aligned}\right\} \therefore u = \frac{\pi a}{2}(x^2 - b^2) \therefore (u) = \frac{\pi a}{2}(c^2 - b^2).$$

Ex. 3. An oblong spheroid.

$$y^2 = \frac{b^2}{a^2}.(2ax - x^2) \therefore du = \pi y^2 dx = \frac{\pi b^2}{a^2} dx.(2ax - x^2)$$

$\therefore u = \frac{\pi b^2}{a^2}\left(ax^2 - \frac{x^3}{3}\right)$ for $c = o$ $\therefore$ the whole solid $=$ $\frac{\pi b^2}{a^2}\left(4a^3 - \frac{8a^3}{3}\right) = \frac{4}{3}\pi b^2 a = \frac{2}{3}$ circumscribing cylinder; and $=$ twice the inscribed cone.

If $b = a$, the spheroid becomes a sphere; or the content of a sphere $= \frac{2}{3}$ circumscribing cylinder.

If the spheroid be *oblate*, then, *mutatis mutandis*, $u = \frac{4}{3}\pi a^2 b$.

Ex. 4. A pyramid.

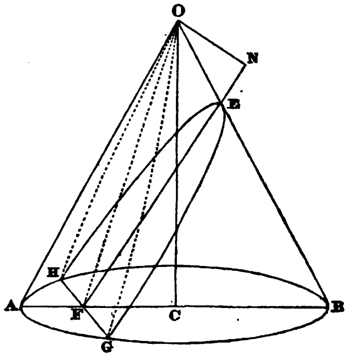
Let ABCD be the base of the pyramid, P its vertex; draw PG perpendicular to ABCD.

Intersect the pyramid by a plane as *abcd* perpendicular to PG or parallel to ABCD, cutting PG in *g*; then the pyramid may be supposed to be generated by *abcd* moving along the axis PG.

Join AG, *ag*; and it may be shown that *abcd* is always similar to ABD.

PG $= a$
P$g = x$
ABCD $=$ A

$\therefore$ (Art. 12. Cor.) $du = abcd \times dx$; but from similar figures, $abcd : \text{ABCD} :: ad^2 : \text{AD}^2 :: \text{P}a^2 : \text{PA}^2 :: \text{P}g^2 : \text{PG}^2 \therefore abcd = \frac{\text{A}x^2}{a^2} \therefore du = \frac{\text{A}}{a^2}.x^2dx$ and $u = \frac{\text{A}}{a^2}\frac{x^3}{3} \therefore$ the whole pyramid $= \frac{\text{A}a^3}{3a^2} = \frac{a\text{A}}{3}$.



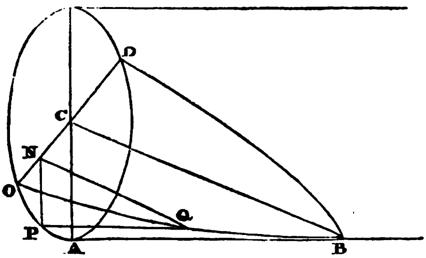
Cor. 1. Hence the content of a cone $= \frac{1}{3}$ circumscribing cylinder; and if a frustum be cut off by a plane which bisects the axis, the frustum : the whole cone :: 7 : 8.

Cor. 2. Hence also may be found the content of an *ungula*, which is a solid cut off from a cone by a given plane.

Let HEG be the given plane, and EHGB the ungula; OC the axis of the cone: draw the plane HOG, and ON perpendicular to FE produced.

The solid BOHG $= \dfrac{\text{BHG} \times \text{OC}}{3}$, and the solid EOHG $= \dfrac{\text{HEG} \times \text{ON}}{3}$, each of which can be calculated; whence their difference which is the ungula can be found.

Ex. 5. Required the content of the ungula of a cylinder made by a section passing through the centre of its base.



Let DBO be the section passing thro' C the centre of the circle perpendicular to the radius AC. In OC take any point N; draw the ordinate NP perpendicular to CO, and from P along the surface of the cylinder, draw PQ parallel to AB meeting the section in Q; join NQ.

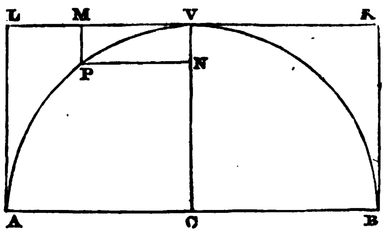
The ungula may be supposed to be generated by the triangle NPQ which is parallel to the triangle CAB, for NP, PQ are each parallel to CA, AB; and the axis is OCD.

AC $= a$
AB $= b$
ON $= x$

From $\triangle$ PQ $=$ PN $\times \dfrac{\text{AB}}{\text{AC}} = \dfrac{b}{a}\sqrt{2ax - x^2}$, . . .

$\therefore\ du = \dfrac{b}{2a}.\ (2ax - x^2)dx$ and $u = \dfrac{b}{2a}\left\{ax^2 - \dfrac{x^3}{3}\right\}$ and the whole solid $= \frac{2}{3}\,a^2b$.

Ex. 6. Required the content of a solid generated by the revolution of a semicircle round a tangent parallel to the diameter of the semicircle.



Let AVB be the semicircle, and let it revolve round the tangent LVK; complete the rectangle ALKB.

Here it will be more convenient to calculate the content of the *vacuum* formed by VAL and VBK, and to subtract these from the cylinder generated by LB.

Draw the ordinate PN and complete the parallelogram VMPN; let V = vacuum formed by VPM $\therefore$
$dV = \odot PM \times d.VM = \odot VN \times d.PN = \pi x^2 \times d.\sqrt{2ax - x^2}$

$\therefore$ integrating by parts, $V = \pi x^2.(2ax - x^2)^{\frac{1}{2}}$
$$-2\pi\int\sqrt{2ax - x^2}.xdx$$
$$= \pi x^2(2ax - x^2)^{\frac{1}{2}} + 2\pi.\int\sqrt{2ax - x^2}(adx - xdx)$$
$$-2\pi.\int\sqrt{2ax - x^2}.adx$$
$$= \pi x^2(2ax - x^2)^{\frac{1}{2}} + \frac{2\pi}{3}.(2ax - x^2)^{\frac{3}{2}} - 2\pi a.\text{VPN, for } C = 0;$$

$\therefore$ vacuum formed by VAL $= \pi a^3 + \dfrac{2\pi a^3}{3} - 2\pi a \times$ VAC
$$= \frac{5\pi a^3}{3} - 2\pi a \times \text{VAC};$$

$\therefore$ the whole vacuum $= \dfrac{10\pi a^3}{3} - 2\pi a \times$ AVB, and cylinder LB $= 2\pi a^3$ $\therefore u = 2\pi a.\text{AVB} - \dfrac{4\pi a^3}{3} = \pi^2 a^3 - \dfrac{4\pi a^3}{3}$.

Ex. 7. Let $y = ae^{-x^2}$; $\therefore (u) = \pi a^3$.

Ex. 8. Required the content of a *groin*.

1. Let the co-ordinate sections be equal semicircles.

ON $= x$	The area of the generating square $= 4\text{NP}^2$
NP $= y$	$= 4(2ax - x^2)$ $\therefore du = 4.(2ax - x^2)dx$ $\therefore u =$
OA $= a$	$4ax^2 - \frac{4}{3}x^3$, and the whole solid $= \frac{8}{3}a^3$.

Whatever be the form of the sections, the content of the groin will be to that of the solid generated by the revolution of the section round its axis as a square to its inscribed circle.

2. The content may be found when the sections on opposite sides of the axis are different curves.

Thus if the two sections are a circle and a parabola whose equations are $y^2 = 2ax - x^2$ and $y^2 = px$, ...
$u = \frac{8}{15}(8a^2\sqrt{2ap} - p^{\frac{1}{2}}.(2a - x)^{\frac{3}{2}}(4a + 3x)$.

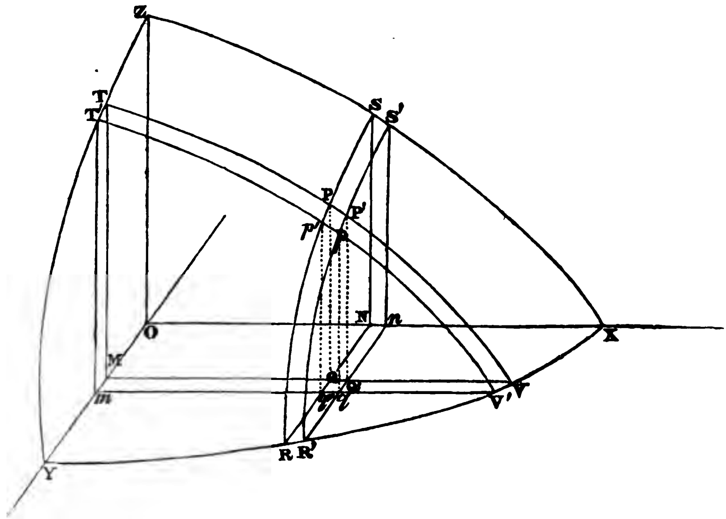
16. We have hitherto represented the fluxions by *finite* lines and areas, according to the method invariably adopted by Sir I. Newton.

In order to ascertain the limiting ratio of evanescing quantities, he constructs finite magnitudes which are always in the same ratio as the evanescing quantities; the last ratio of these finite magnitudes is evidently the ratio required. In some of the following propositions, for the sake of con-

ciseness, we shall represent the fluxions by the small increments whose ratio is to be taken in the limit; this changes the scale on which dx is taken, but will not affect the accuracy of the result.

Upon the same principle of convenience, solids and areas may be compared with lines, though they are heterogeneous quantities. The area of a rectangle whose sides are 3 and 4, when numerically represented, may be considered as bearing a certain ratio to either of its sides, though in one case the numbers represent superficial, and in the other linear units.

17. *Required the content of a solid bounded by three planes, each of which is perpendicular to the other two, and by a curved surface whose equation is known.*



Let u be that part of the solid which is bounded by OZ and by two planes SPRN, TPVM respectively parallel to ZY and ZX, and intersecting in the line PQ; then since u is bounded by the four planes OS, SQ, QO, OT, and the surface ZP, it is necessarily an implicit function of x and y, and its whole increment arises from the variation both of x and of y.

Take Nn, Qq' small and cotemporary increments of ON, NQ; through n and q' draw the planes S$'$P$'$R$'$, T$'p'$V$'$ parallel

to SPR, TPV intersecting in pq; and join PQ, P'Q', $p'q'$, pq, which together with the surface Pp form a curvilinear prism whose base is Qq.

Now $\frac{du}{dx}dx : dx =$ the limit of SNQ'P : Nn; or, representing the fluxions by the increments for the sake of brevity, $\frac{du}{dx}dx =$ SNQ'P in the limit, and consequently

$$\frac{d\left(\frac{du}{dx}dx\right)}{dy}dy =$$ the ultimate value of the increment of SNQ'P arising solely from the variation of $y =$ the ultimate value of the prism P$q =$ Qq × PQ; *i.e.* $\frac{d^2u}{dxdy}dxdy = zdxdy$, and reversing the operations indicated by the symbols, $u = \iint zdxdy$.

If we reverse the order in which the partial fluxions of u are taken, there will result the same paralleleliped Qp to represent its second partial fluxion, and consequently the order in which the integrations are taken will not affect the result; or $u = \iint zdxdy = \iint zdydx$.

18. The demonstration of the preceding article may be made to depend upon Taylor's theorem.

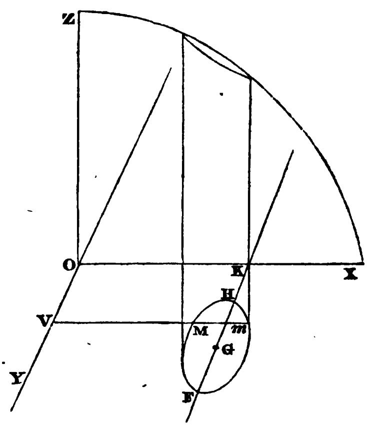
For on the base Qq describe four parallelelipeds with altitudes PQ, P'Q', $p'q'$ and pq; then, whatever be the values of h and k, it is manifest that the magnitude of the prism lies between the magnitudes of the parallelelipeds, being greater than the least and less than the greatest. But, since the prism is a second partial increment of the solid, the first term of its developement is $\frac{d^2u}{dxdy}hk$. Also the parallelelipeds when developed are $hk.z$, $hk\left\{z + \frac{dz}{dx}\frac{h}{1} + \&c.\right\}$, $hk\left\{z + \frac{dz}{dy}\frac{k}{1} + \&c.\right\}$, and $hk\left\{z + \frac{dz}{dx}\frac{h}{1} + \frac{dz}{dy}\frac{k}{1} + \&c.\right\}$; wherefore (3. 3) $\frac{d^2u}{dxdy} = z$ and $u = \iint zdxdy$.

19. Since $\frac{d^2u}{dydx}dydx = zdydx = dyzdx$, therefore .... $\frac{du}{dy}dy = dy\int zdx$; but z is a function of x and y given by the equation of the surface, and we have to integrate $\int zdx$ on the supposition that x alone varies; and consequently, the correction of this fluent must be a sole function of y; represent it by Y', then $\frac{du}{dy}dy = dy\left\{\int zdx + \mathrm{Y}'\right\}$; and integrating a second time, and representing $\int \mathrm{Y}'dy$ by Y, we have $u = \mathrm{X} + \mathrm{Y} + \iint zdxdy$ where X is a sole function of x.

In the first integration the value of $\int zdx$ must be taken for the whole extent of the solid, y being considered as constant; in the preceding diagram, it must be calculated be tween $x = 0$ and $x = \mathrm{MV}$, and the result being a function of y, call it P; we shall then have to integrate $\int \mathrm{P}dy$ between the values $y = 0$ and $y = \mathrm{OY}$.

If u is to be included between the two values $x = \mathrm{A}$, $x = a$ and $y = \mathrm{B}$, $y = b$, find the value of $\int zdx$ between $x = \mathrm{A}$, $x = a$, y being constant, let this $= \mathrm{P}$; and then integrate $\int \mathrm{P}dy$ between $y = \mathrm{B}$, $y = b$.

Next suppose that the part to be calculated is as much of the solid as is included in a cylinder whose base coincides with XY, and whose axis is parallel to Z.



Let the base of the cylinder be a circle; draw $\mathrm{VM}m$ parallel to OX cutting the circumference in M and m. In this case, we shall have to integrate $\int zdx$ between the limits $x = \mathrm{VM}$, $x = \mathrm{V}m$; and the position and magnitude of the circle being given, the fluent will be obtained in terms of OV or y. Draw the diameter FGHK perpendicular to OX, meeting OX in K; and the second fluent must be integrated between $y = \mathrm{KH}$, $y = \mathrm{KF}$.

The same method is applicable when the base of the cylinder is any known curve.

Cor. 1. This method of calculating u follows immediately from the principle that the fluxion of a solid is as the generating plane and its velocity conjointly.

Cor. 2. u may be put under the form of a triple fluent. For $u = z\iint dxdy = \iiint dxdydz$. An area, which $=\int ydx = \iint dxdy$, may be considered as a double, and a line as a single fluent.

20. In this article we shall give two examples to show that in the integration of double fluents, the result is not affected by the order of the integrations. In these examples x and y may be independent quantities, as they are not connected by an equation of condition.

Ex. 1. $u = \iint \dfrac{dxdy}{ax+by}$.

First considering y constant, we have

$$\frac{\int dx}{ax+by} = \int dx \left\{ \frac{1}{ax} - \frac{by}{a^2x^2} + \frac{b^2y^2}{a^3x^3} - \&c. \right\} + \text{Y}'$$

$$= \frac{1}{a} lx + \frac{by}{a^2x} - \frac{b^2y^2}{2a^3x^2} + \&c. + \text{Y}'$$ where Y' is an arbitrary function which cannot be determined, since there is no equation of condition. Hence u, which . . . .

$$= \int dy \int \frac{dx}{ax+by}, \quad \frac{y}{a}\, lx + \frac{by^2}{2a^2x} - \frac{b^2y^3}{2.3a^3x^2} + \&c. + \text{Y} + \text{X}.$$

Next integrate u, first considering x constant; then we have

$$\int \frac{dy}{ax+by} = \frac{y}{ax} - \frac{by^2}{2a^2x^2} + \frac{b^2y^3}{3a^3x^3} - \&c. + \text{X}',$$ and integrating again, $$u = \int dx \int \frac{dy}{ax+by} = \frac{y}{a}\, lx + \frac{by^2}{2a^2x} - \frac{b^2y^3}{2.3a^3x^2} + \&c. \ldots$$

$+ \text{X} + \text{Y}$, which is the same result as before; for X and Y being arbitrary functions, may be assumed the same in the two cases.

Ex. 2. $u = \iint \dfrac{dxdy}{x^2+y^2}$.

$$\int \frac{dx}{x^2+y^2} = \frac{1}{y} \tan.^{-1} \frac{x}{y} + \text{Y}' \therefore$$

$$u = \int \frac{dy}{y} \tan.^{-1} \frac{x}{y} + \text{Y} + \text{X}$$

$$=\int dy\left\{\frac{x}{y^2}-\frac{x^3}{3y^4}+\frac{x^5}{5y^6}-\&c.\right\}+\text{Y}+\text{X}\ (4.\ 8.\ \text{Ex. } 7)$$

$$=\text{X}+\text{Y}-\frac{x}{y}+\frac{x^3}{3^2y^3}-\frac{x^5}{5^2y^5}+\&c.$$

Again $\int\frac{dy}{x^2+y^2}=\frac{1}{x}\tan.^{-1}\frac{y}{x}+\text{X}'$; but since $\tan.^{-1}\frac{y}{x}$ $+\tan.^{-1}\frac{x}{y}=\frac{\pi}{2}\ \therefore \tan.^{-1}\frac{y}{x}=\frac{\pi}{2}-\left\{\frac{x}{y}-\frac{x^3}{3y^3}+\frac{x^5}{5y^5}-\&c.\right\}$

$$\therefore\frac{1}{x}\tan.^{-1}\frac{y}{x}=\frac{\pi}{2x}-\frac{1}{y}+\frac{x^2}{3y^3}-\frac{x^4}{5y^5}+\&c.$$

$$\therefore u=\frac{\pi lx}{2}-\frac{x}{y}+\frac{x^3}{3^2y^3}-\frac{x^5}{5^2y^5}+\&c.+\text{X}+\text{Y}$$

$=\text{X}+\text{Y}-\frac{x}{y}+\frac{x^3}{3^2y^3}-\frac{x^5}{5^2y^5}+\&c.$ since $\frac{\pi lx}{2}$ may be included in X.

21. *Examples.*

Ex. 1. Let O be the centre of a sphere; required the content of ZOXY.

The equation is $x^2+y^2+z^2=r^2\ \therefore z=\sqrt{r^2-y^2-x^2}$ $\therefore\frac{du}{dy}=\int dx\sqrt{r^2-y^2-x^2}=$ cir. area, rad. $=\ldots\ldots$ $\sqrt{r^2-y^2}$, absc. $=x$, $+$ C; of which the limits are $x=0$ and $x=\text{MV}=\sqrt{r^2-y^2}\ \therefore\left(\frac{du}{dy}\right)=\frac{\pi}{4}(r^2-y^2)\ \ldots\ldots$ $\therefore u=\frac{\pi}{4}\left(r^2y-\frac{y^3}{3}\right)\therefore(u)=\frac{\pi r^3}{6}$, and the whole sphere $=\frac{8\pi r^3}{6}=\frac{4\pi r^3}{3}$.

If we integrate the first fluent between the values $x=\sqrt{r^2-y^2}$, $x=-\sqrt{r^2-y^2}$; and the second between $y=r$, $y=-r$, the result will give the whole hemisphere above the plane XY.

Ex. 2. Let the equation be $\frac{x}{a}+\frac{y}{b}=\frac{\sqrt{a^2-z^2}}{a}\ \ldots\ldots$ (7. 46. Ex. 3).

Here $z^2 = a^2 - \left(x + \frac{ay}{b}\right)^2 =$, by substitution, $a^2 - v^2$,

$\therefore \int z dx = \int dv \sqrt{a^2 - v^2} =$ cir. area, rad. $= a$, absc. $= v + c$;

and integrating this between $x = 0$, $x = a - \frac{ay}{b}$, *i. e.* between $v = \frac{ay}{b}$, $v = a$, we have $\left(\frac{du}{dy}\right) = \frac{\pi a^2}{4} -$ cir. area, rad. $= a$, absc. $= \frac{ay}{b} \therefore u = \frac{\pi a^2 y}{4} - y \times$ cir. area. rad. $= a$, absc. $= \frac{ay}{b}$, $+ \int \frac{aydy}{b} \sqrt{a^2 - \frac{a^2y^2}{b^2}} + \text{x}$; and integrating between $y = 0$, $y = b$, there results $(u) = \frac{a^2 b}{3}$.

The same result will be obtained by considering the solid to be generated by a triangle whose plane is parallel to XY.

Ex. 3. An *ellipsoid.*

The equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (7. 46. Ex. 6.) $\therefore$

$z = \frac{c}{a} \sqrt{\frac{a^2(b^2 - y^2)}{b^2} - x^2} \therefore \int z dx = \frac{c}{a}$ cir. area, rad. . . . $= \frac{a}{b} \sqrt{b^2 - y^2}$ absc. $= x$, $+ \text{Y}$; and here MV $= \frac{a}{b} \sqrt{b^2 - y^2}$

$\therefore \left(\frac{du}{dy}\right) = \frac{c}{a} \cdot \frac{\pi a^2}{4b^2}(b^2 - y^2) \therefore u = \frac{\pi ac}{4b^2}\left(b^2 y - \frac{y^3}{3}\right) + \text{x}$

$\therefore (u) = \frac{\pi abc}{6}$ and the whole solid $= \frac{4}{3}\pi abc$.

Ex. 4. An *elliptick paraboloid.*

The equation is $\frac{x^2}{a} + \frac{y^2}{b} = z \therefore \int z dx = \frac{x^3}{3a} + \frac{y^2 x}{b}$.

Let the axis of the solid $= c$; then $\text{OX}^2 = ac$, $\text{OY}^2 = bc$

and to find MV, we have $\frac{\text{MV}^2}{ac} + \frac{y^2}{bc} = 1 \therefore \text{MV} = \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} \sqrt{bc - y^2}$

$$\therefore \left(\frac{du}{dy}\right) = \frac{a^{\frac{1}{2}}(bc - y^2)^{\frac{3}{2}}}{3b^{\frac{3}{2}}} + \frac{a^{\frac{1}{2}}y^2(bc - y^2)^{\frac{1}{2}}}{b^{\frac{3}{2}}}$$

$$= \frac{a^{\frac{1}{2}}c(bc - y^2)^{\frac{1}{2}}}{3b^{\frac{1}{2}}} + \frac{2a^{\frac{1}{2}}y^2(bc - y^2)^{\frac{1}{2}}}{3b^{\frac{3}{2}}}$$

$$\therefore u = \frac{a^{\frac{1}{2}}c}{3b^{\frac{1}{2}}} \text{ cir. area rad.} = \sqrt{bc}, \text{ absc.} = y, \ldots\ldots$$

$$+ \frac{2a^{\frac{1}{2}}}{3b^{\frac{3}{2}}}\left\{ -\frac{y(bc - y^2)^{\frac{3}{2}}}{4} + \frac{bc}{4}\int dy \sqrt{bc - y^2} \right\} \ldots\ldots$$

$$= \frac{a^{\frac{1}{2}}c}{2b^{\frac{1}{2}}} \text{ cir. area rad.} = \sqrt{bc}, \text{ absc.} = y, - \frac{a^{\frac{1}{2}}y(bc - y^2)^{\frac{3}{2}}}{6b^{\frac{3}{2}}}$$

$\therefore (u) = \dfrac{a^{\frac{1}{2}}c}{2b^{\frac{1}{2}}} \cdot \dfrac{\pi bc}{4} = \dfrac{\pi a^{\frac{1}{2}}b^{\frac{1}{2}}c^2}{8}$ and the whole solid corresponding to the axis $c = \dfrac{\pi a^{\frac{1}{2}}b^{\frac{1}{2}}c^2}{2}$.

Ex. 5. Required the space included between the base of a cylinder and the surface of a cone whose vertex is in the centre of the base and its axis perpendicular to it.

The equation is $z = t\sqrt{x^2 + y^2}$ (7. 45) where $t = \tan. \angle$ which a side of the cone makes with the base of the cylinder.

$\int dx \sqrt{x^2 + y^2} = \frac{1}{2}x\sqrt{x^2 + y^2} + \dfrac{y^2}{2}\, l\, \dfrac{x + \sqrt{x^2 + y^2}}{y}$ (2. 55. Ex. 1), and integrating this to $x = \sqrt{r^2 - y^2}$, $\left(\dfrac{du}{dy}\right) \ldots\ldots$

$$= \frac{r}{2}\sqrt{r^2 - y^2} + \frac{y^2}{2}\, l\, \frac{r + \sqrt{r^2 - y^2}}{y},$$

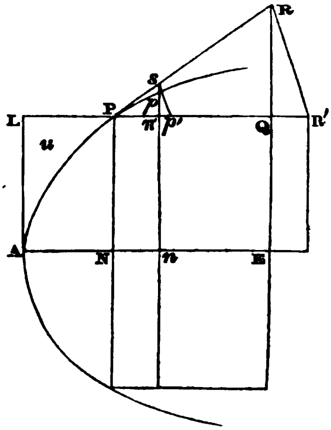
$$\therefore \frac{u}{t} = \frac{r}{2}\int dy \sqrt{r^2 - y^2} + \frac{y^3}{6}\, l\, \frac{r + \sqrt{r^2 - y^2}}{y} + \int \frac{y^3}{6} \cdot \frac{r\,dy}{y\sqrt{r^2 - y^2}}$$

$$= \frac{r}{2}\int dy \sqrt{r^2 - y^2} + \frac{y^3}{6}\, l\, \frac{r + \sqrt{r^2 - y^2}}{y} + \frac{r}{6} \ldots\ldots$$

$\left\{-y\sqrt{r^2-y^2}+\int dy\sqrt{r^2-y^2}\right\} \therefore \frac{(u)}{t} = \frac{4r}{6} \cdot \frac{\pi r^2}{4}$ or the whole space $= \frac{2}{3}\pi r^2 . tr = \frac{2}{3}$ cylinder; which is the same result as that obtained by subtracting the cone from the cylinder.

If the base of the cylinder is any known curve, or the solid which is inserted in it be any known solid, the included space may be calculated by the same method.

22. *Required the surface of a solid of revolution.*



Let AP be the curve which generates the proposed surface; and the same construction remaining as in Art. 1, draw the fluxional triangle PQR. Produce np to meet the tangent in S; and in PQ take Pp' always = Ps and PR$'$ = PR; join sp' and RR$'$ which are therefore parallel, and consequently Pp' : PR$'$: : Ps : PR : : Pπ : PQ.

The increment of the surface generated by AP : the cotemporary increment of the surface generated by LP : : increment generated by Pp : increment generated by Pπ; but the quantity of surface generated by a revolving line manifestly varies as the line and its distance from the axis jointly; and in the limit Pp = PS = Pp'; also the distance of these three lines from AN is the same, viz. PN; hence in the limit the surface generated by Pp' may be substituted for the surface generated by Pp, and consequently d. surface AP : d. surface LP : : surface generated by Pp' : surface generated by Pπ : : surface generated by PR$'$: surface generated by PQ; but the surface generated by PQ is the fluxion of the surface LP on the scale that PQ = d. AN; hence, on the same scale, d. surface AP = surface generated by PR$'$ = circumference of circle NP × PR$'$, or $du = 2\pi y ds$ and $u = 2\pi \int y ds$.

Cor. The fluxion of the surface is as the generating circumference and its velocity jointly.

23. *Required the surface of a solid of revolution by means of Taylor's theorem.*

It is manifest that the surface generated by Pp is always greater than if it were extended in the right line PQ, and less than if it revolved at a distance np: or U $-$ u is less than $2\pi \,.\, np \times$ Pp, and greater than $2\pi \,.\,$ NP $\times$ Pp, and therefore by Taylor's theorem $2\pi(y+k)\left\{\frac{ds}{dy}\frac{k}{1}+\frac{d^2s}{dy^2}\frac{k^2}{1.2}+\&c.\right\}$, $\frac{du}{dy}\frac{k}{1}+\frac{d^2u}{dy^2}\frac{k^2}{1.2}+\&c.$ and $2\pi y\left\{\frac{ds}{dy}\frac{k}{1}+\frac{d^2s}{dy^2}\frac{k^2}{1.2}+\&c.\right\}$ are always in the order of their magnitudes, and consequently (3. 3.) we have $\frac{du}{dy}=\frac{2\pi yds}{dy}$, or $u = 2\pi\int yds$.

24. *Examples.*

Ex. 1. Required the surface of a sphere.

$ds : dx :: a : y \therefore 2\pi yds = 2\pi adx$ and the whole surface $= 2\pi a \,.\, 2a = 4\pi a^2 = 4$ areas of a great circle of the sphere.

Cor. The surface of a spherical segment $= 2\pi ax = \pi .2ax = \pi \,.\, (\text{chord})^2 =$ the area of a circle whose radius is the distance of the pole from the circumference of its base.

Ex. 2. Paraboloid.

$$y^2 = ax \therefore dx=\frac{2ydy}{a} \therefore ds=\sqrt{dx^2+dy^2}=dy\sqrt{\frac{4y^2}{a^2}+1}$$

$$= dy\frac{\sqrt{4y^2+a^2}}{a} \therefore du = 2\pi yds = \frac{2\pi}{a}\,.\sqrt{4y^2+a^2}\,.ydy$$

$$\left.\begin{array}{l}\therefore u = \frac{2\pi}{12a}.(4y^2+a^2)^{\frac{3}{2}}+c \\ o = \frac{2\pi}{12a}.a^3+c\end{array}\right\}\therefore u = \frac{\pi}{6a}\left\{(4y^2+a^2)^{\frac{3}{2}} - a^3\right\}.$$

Ex. 3. The surface generated by a cycloid revolving round its base.

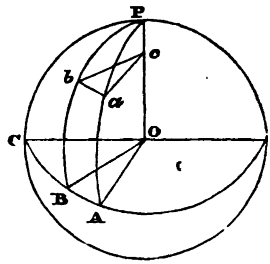
Let V be the vertex and AC the semibase, draw PN perpendicular to the base.

The surface may be supposed to be generated by the circle PN, hence $du = 2\pi \,.\,$ PN $\times\, d \,.\,$ VP . . . .

$$= 2\pi \,.\, (a - x) \times \frac{a^{\frac{1}{2}}dx}{x^{\frac{1}{2}}} \therefore u = 2\pi a^{\frac{1}{2}}(2ax^{\frac{1}{2}} - \tfrac{2}{3}x^{\frac{3}{2}}) \text{ which is}$$

the surface generated by VP, hence whole surface $= \frac{16\pi a^2}{3}$.

Ex. 4. If P be the pole of the great circle ABC, O being the centre of the sphere and OB, OA joined; the surface PAB shall = twice the sector AOB.



Suppose PAB to be generated by the intersection of a plane acb moving on PO parallel to AOB.

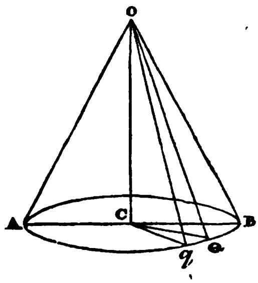
$$\left.\begin{array}{l} PO = r \\ Oc = x \\ cb = y \\ Pa = s \\ AB = k \end{array}\right| \begin{array}{l} \text{then } r : y :: k : \dfrac{ky}{r} = \text{arc } ab \therefore \dfrac{kyds}{r} = d\,.\,\text{surface} \\ Pab \text{ and } r : y :: ds : -dx \therefore ds = -\dfrac{rdx}{y} \\ \therefore du = -kdx \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

$$\left.\begin{array}{l} \therefore u = -kx + c \\ \text{and } 0 = -kr + c \end{array}\right\} \therefore u = k\,.(r - x) \text{ and whole surface}$$

$= kr =$ twice OAB, and the whole surface of the sphere = 4 areas of its great circle.

Ex. 5. The surface of a cone : the area of its base : : the hypothenuse of the generating triangle : its base.

For take OQ, Oq two positions of the hypothenuse near to each other; join CQ, Cq, C being the centre of the base; then d. surface BOQ : d. area BCQ : : the limit of OQq : CQq : : OQ : CQ which is a constant ratio, and consequently the whole fluents generated are in the same ratio.



Cor. Hence the surface AOB $= \dfrac{\pi b^2}{\cos. \angle B}$ where b = radius of base.

Ex. 6. A spheroid.

$$ds = dx\frac{\sqrt{a^2 - e^2x^2}}{\sqrt{a^2 - x^2}} \therefore du = 2\pi y ds = \frac{2\pi b}{a}\sqrt{a^2 - e^2x^2}\,.\,dx \therefore$$

$u = \dfrac{2\pi b}{ea} \times$ cir. area, rad. $= a$, absc. $= ex$, hence the whole surface $= \dfrac{4\pi b}{ea} \times$ cir. area, rad. $= a$, absc. $= ea$, or $= \dfrac{4\pi ab}{e}$ $\times$ cir. area, absc. $= e$.

Cor. When $a = b$, the spheroid becomes a sphere, and the value of u becomes $\frac{0}{0}$; but since the circular area is ultimately equal to a parallelogram, whose base is e and height $= 1$, $(u) = \frac{4\pi a^2.e}{e} = 4\pi a^2$.

Ex. 7. A groin. Vid. Art. 15. Ex. 8.
The fluxion of the surface generated by $2\text{NP} = 2\text{NP} \times d\text{AP}$ $= 2yds = 2adx \therefore du = 8adx$, and $u = 8ax$, and the whole surface $= 8a^2$.

25. PRAXIS.

1. The surface generated by a cycloid revolving round its axis $= 2\pi^2 a^2 - \frac{8}{3}\pi a^2$.
2. The surface generated by a cycloid revolving round a tangent at the vertex $= \frac{16}{3}$ of the generating circle.

26. *Required the surface of a solid bounded by three planes, each of which is perpendicular to the other two.*

The same construction remaining as in Art. 17, suppose the tangent plane to be drawn at the point P, and let α be the angle at which it is inclined to the plane XY, then

$$(7.\ 52.)\ \cos.\alpha = \sqrt{\frac{1}{1 + \frac{\text{C}^2}{\text{A}^2} + \frac{\text{C}^2}{\text{B}^2}}} = \frac{1}{\sqrt{1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}}} =$$

by substitution $\frac{1}{\sqrt{1+p^2+q^2}}$, and since in the limit the sides of $\text{PP}'pp'$ may be substituted for the corresponding lines in the tangent plane, and that $\text{QQ}'qq'$ is its orthographick projection on XY, we have $\text{PP}'pp' = \text{QQ}'qq' \times \frac{1}{\cos.\alpha}$.

But it may be shown as in Art. 17, that if $u =$ the required surface, $\frac{d^2u}{dxdy}dxdy =$ the limit of $\text{PP}'pp'$ on the scale that $dx = \text{N}n$; whence $\frac{d^2u}{dxdy}dxdy = \frac{\text{QQ}'qq'}{\cos.\alpha} = \frac{dxdy}{\cos.\alpha}$, and

$$u = \iint \frac{dxdy}{\cos.\alpha} = \iint dxdy\sqrt{1 + p^2 + q^2}.$$

This article may be made to depend upon Taylor's theorem in the same manner as Art. 18, by showing that

the developement of $\overline{PP'pp'}$ lies between two series, whose first terms are $hk\sqrt{1+p^2+q^2}$. Lacroix has given the investigation in a note, tome 2. c. 2. 523.

27. *Examples.*

Ex. 1. Required the surface of a plane intercepted by the rectangular axes.

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \therefore p=-\frac{c}{a},\ q=-\frac{c}{b} \therefore \sqrt{1+p^2+q^2}$$

$$=c\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} \therefore u=c\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}\iint dxdy.$$

$\int dx = x + \text{Y}$, which is to be integrated between $x=0$,

$$x=\frac{a}{b}(b-y) \therefore u=\frac{ac}{b}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}\int dy(b-y)\ldots$$

$$=\frac{ac}{b}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}:(by-\tfrac{1}{2}y^2) \therefore (u)\ldots\ldots\ldots$$

$$=\frac{abc}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}=\tfrac{1}{2}\sqrt{a^2b^2+a^2c^2+b^2c^2}.$$

Ex. 2. A sphere.

$$x^2+y^2+z^2=r^2 \therefore p=-\frac{x}{z},\ q=-\frac{y}{z}\ldots\ldots$$

$$\therefore u=\iint dxdy\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}=\iint\frac{rdxdy}{z}\ldots\ldots\ldots$$

$$=\iint\frac{rdxdy}{\sqrt{r^2-x^2-y^2}}.$$

$\int\frac{dx}{\sqrt{r^2-y^2-x^2}}=\sin.^{-1}\frac{x}{\sqrt{r^2-y^2}}+\text{Y}$, which integrated from $x=0$ to $x=\sqrt{r^2-y^2}$, $=\frac{\pi}{2} \therefore u=\frac{\pi r}{2}\int dy=\frac{\pi ry}{2}+\text{X}$

$\therefore (u)=\frac{\pi r^2}{2}$, and the whole surface $=\frac{8\pi r^2}{2}=4\pi r^2$.

Ex. 3. A right cone.

$$z=t\sqrt{x^2+y^2} \therefore p=\frac{tx}{\sqrt{x^2+y^2}},\ q=\frac{ty}{\sqrt{x^2+y^2}}\ldots\ldots$$

$$\therefore \sqrt{1+p^2+q^2}=\sqrt{1+t^2} \therefore u=\sqrt{1+t^2}\iint dxdy.$$

$\int dx = x + \text{Y}$, and this is to be integrated between $x = 0$, $x = \sqrt{b^2 - y^2}$, if b = radius of base $\therefore u = \sqrt{1+t^2}.\int dy \sqrt{b^2-y^2}$

$\therefore (u) = \sqrt{1+t^2}\,\frac{\pi b^2}{4}$, and the whole surface $= \sqrt{1+t^2}.\pi b^2$.

Vid. 24. Ex. 5.

Ex. 4. Two equal and opposite parabolas are described in the plane XY, touching at the vertex which is placed in the origin, their axes coinciding with X. A right cone is also described whose vertex is the origin, and whose axis coincides with Z; required to calculate as much of the surface of the cone as is included in a cylinder whose base is the parabolick curves and axis Z, and a plane drawn parallel to and at a given distance from XZ.

Let $y^2 = ax$ be the equation of the parabolas.
b = the given distance.
$u = \sqrt{1+t^2}\iint dx dy$ as in the preceding example.

Also $\int dx$ is in this case to be integrated between $x = 0$,

$$x = \frac{y^2}{a} \therefore u = \sqrt{1+t^2}\int \frac{y^2 dy}{a} = \sqrt{1+t^2}\,\frac{y^3}{3a} + \text{X} \ldots .$$

$\therefore (u) = \sqrt{1+t^2}\,\frac{b^3}{3a}$; and the whole surface between the given plane and XZ $= \sqrt{1+t^2}\,\frac{2b^3}{3a}$.

There are very few instances in which these double fluents can be integrated, except by certain transformations or other artifices of calculation, which will be given in the second volume.

28. *Required to deduce the formulæ for the content and the surface of a solid of revolution from Arts.* 17 and 26.

When the body is a solid of revolution, every section passing through the axis of revolution is the generating curve, and every section perpendicular to the axis is a circle. In fig. 17 the section passing through P, if OX is the axis of revolution, is a circle whose radius = NR =, by substitution, X, where X is a sole function of x determined by the equation of the generating curve. Also, from the property of the circle, $z = \sqrt{\text{X}^2 - y^2}$; whence $\int z dy = \int dy \sqrt{\text{X}^2 - y^2}$ =, integrating on the supposition that x is constant, cir. area, rad. = X, absc. $= y + \text{C}$; and if this be integrated between $y = -\text{X}$, $y = +\text{X}$, there

results $\int z dy = \frac{\pi x^2}{2}$ for the whole solid above the plane XY, and consequently the whole fluent of $z dy = \pi x^2 = \pi y^2$ if y is the ordinate of the solid of the revolution; and u, which $= \iint z dy dx, = \int \pi y^2 dx$.

Next for the surface; $p = \frac{dz}{dx} = (3.\ 8.) \frac{x}{\sqrt{x^2 - y^2}} \frac{dx}{dx} \ldots$

and $q = \frac{dz}{dy} = \frac{-y}{\sqrt{x^2 - y^2}}$, therefore $1 + p^2 + q^2 = 1 \ldots$

$+ \frac{1}{x^2 - y^2}\left(\frac{x^2 dx^2}{dx^2} + y^2\right) = \frac{x^2}{x^2 - y^2} \cdot \left(1 + \frac{dx^2}{dx^2}\right)$, and $\ldots$

$\int dy \sqrt{1 + p^2 + q^2} = \sqrt{1 + \frac{dx^2}{dx^2}} \cdot \int \frac{x dy}{\sqrt{x^2 - y^2}} =$, integrating between $y = -x$, $y = +x$, $\sqrt{1 + \frac{dx^2}{dx^2}} \pi x$; $\ldots$

or $u = \int 2\pi x dx \sqrt{1 + \frac{dx^2}{dx}} = 2\pi \int x dx \sqrt{1 + \frac{dx^2}{dx^2}} \ldots$

$= 2\pi \int y dx \sqrt{1 + \frac{dy^2}{dx^2}}$, which is the formula of Art. 22.

CHAPTER X.

Spirals.

1. *Def.* 1. A curve which is generated by a variable line revolving in the same plane about a fixed point is called a *spiral*.

The spiral is, in general, determined and its properties ascertained from the relation which subsists between the revolving line, which is called the *radius vector*, and the angle described.

Let SP the radius vector $= r$, (vid. fig. 4) $\angle$ ASP $= \theta$, rad. $= 1$.

With radius SA describe a circle AD cutting SP in D; then AP may be supposed to be generated by an ordinate as DP, which produced always passes through S, moving along the abscissa AD; or SP, which $=$ SD $+$ DP, may be considered as the function and AD its independent variable.

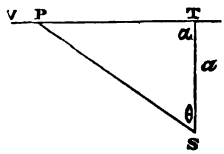
Def. 2. S is the *pole* of the spiral, and the equation which expresses the relation between SP and the $\angle$ ASP is called its *polar equation*. SP and AD are the *polar co-ordinates*.

Some writers characterize the spiral by the equation between SP and a perpendicular drawn from S upon a tangent at P.

2. *Examples.*

Ex. 1. Let AP be the Apollonian parabola, A the vertex, its focus being the pole, SA $= a$, SY perpendicular on the tangent at P $= p$, then $r = 2a - r\cos.\theta$ and $p^2 = ar$ are its two equations, the first of which is its *polar* equation.

Ex. 2. A right line as TV may be considered as a spiral.



For take S a known point round which SP revolves, so that P describes the line TPV; let the fixed $\angle$ T $= \alpha$, $\angle$ PST $= \theta$; $\therefore$ ST : SP :: $\sin.(\alpha + \theta)$: $\sin.\alpha$; or $r = \dfrac{a.\sin.\alpha}{\sin.(\alpha + \theta)}$ is the polar equation to a right line considered as a spiral.

With radii SP, SA describe the circular arcs PV, AD; draw PQ a tangent to the circle at P or perpendicular to SP, and take it to represent the fluxion of VP; draw QR at right angles to PQ meeting a tangent to the spiral at P in R; and draw Sπps near to SP; then, since $sp\pi$ is ultimately parallel to QR, it may be shown as in Ch. 8. Art. 2, that the sides of the curvilinear figure Pπp are ultimately in the ratio of the sides of the triangle PQR; whence PR and QR represent the fluxions of AP and SP on the scale that PQ = the fluxion of VP.

Let SA = 1	
AP = s	then VP = SP.AD = $r\theta$; and we have
SP = r	PR = ds, QR = dr, and PQ = $r \times d\theta$.
$\angle$ ASP = θ	

Cor. If the $\angle$ ASD be supposed to increase uniformly, PQ the base of the fluxional triangle must be made to vary as the radius vector.

6. *Required to calculate the subtangent and the subnormal.*

Def. Through S draw T′SN′ perpendicular to SP, meeting the tangent in T′ and the normal in N′; then ST′ is the *subtangent*, and SN′ the *subnormal.*

First, to calculate the subtangent.

From $\triangle^s$ PST′ PQR; $ST' = \dfrac{SP \times PQ}{QR} = \dfrac{r^2 d\theta}{dr}$ = (in Lagrange's notation) $\dfrac{r^2}{r'}$.

Next, to calculate the subnormal.

From $\triangle^s$ PSN′, PQR; $SN' = \dfrac{SP \times QR}{PQ} = \dfrac{dr}{d\theta} = r'$.

From which expressions the values of the subtangent and subnormal may be calculated, if the polar equation be given.

Cor. Hence tangent $PT' = \sqrt{r^2 + \dfrac{r^4}{r'^2}} = \dfrac{r}{r'}\sqrt{r^2 + r'^2}$,

and normal $PN' = \sqrt{r^2 + r'^2}$.

7. *Required the perpendicular on the tangent in terms of the radius vector and the angle described.*

Let SY $= p$; then $\triangle^s$ SPY, PQR; $SY = \dfrac{SP.PQ}{PR} = \ldots\ldots$

$\dfrac{r^2 d\theta}{\sqrt{dr^2 + r^2 d\theta^2}}$; or $p = \dfrac{r^2}{\sqrt{r^2 + r'^2}}$.

8. *Required to draw the asymptote of a spiral.*

Find the value of θ when r is infinite; this determines the ultimate position of SP; then find the corresponding value either of ST′ or of SY; if these be finite, the spiral admits of an asymptote and its position is determined.

9. *Required to calculate the angle which the radius vector makes with the curve.*

$$\text{Tan.} \angle \text{SPY} = \text{tan.} \angle \text{PRQ} = \frac{\text{PQ}}{\text{QR}} = \frac{rd\theta}{dr} = \frac{r}{r'}.$$

$$\text{Or thus; sin.} \angle \text{SPY} = \frac{p}{r} = \frac{r}{\sqrt{r^2 + r'^2}}.$$

Cor. If the spiral meets the pole, the ultimate value of these functions when r is diminished without limit will give the angle which the *initial* direction of the curve makes with the axis.

10. *Required the length of any part of a spiral.*

Since $\text{PR}^2 = \text{PQ}^2 + \text{QR}^2$, therefore $s = \int \sqrt{dr^2 + r^2 d^2\theta^2}$ $= \int dr \sqrt{1 + \left(\frac{r}{r'}\right)^2}$; which can be calculated if the equation is given.

11. *Required the area of a spiral.*

Join SR, SQ; then inc. ASP : inc. AP :: S P p : P p :: ultimately, SPs : Ps :: Δ SPR : PR; hence, on the scale that PR = the fluxion of AP, the fluxion of ASP = Δ SPR = Δ SPQ.

Hence $du = \dfrac{\text{SP} \times \text{PQ}}{2} = \dfrac{r^2 d\theta}{2}$, and $u = \frac{1}{2}\int r^2 d\theta$.

Cor. 1. The fluxion of ASP = the fluxion of VSP.

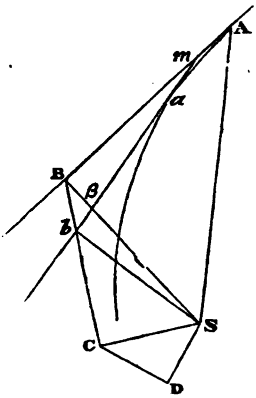
Cor. 2. The fluxion of ASP also = Δ SPR = $\dfrac{\text{PR} \times \text{SY}}{2} = \dfrac{pds}{2}$, and $u = \frac{1}{2}\int pds$.

In a circle, $p = r$; therefore $du = \dfrac{rds}{2}$ and $u = \dfrac{rs}{2}$, and the whole area $= \dfrac{r}{2} \times$ the circumference.

Otherwise. Draw PN perpendicular to ASN, then since u = SAPNS − Δ PNS, therefore $du = ydx - \dfrac{ydx + xdy}{2} =$

$\frac{ydx - xdy}{2}$; but $\frac{y}{x} = -\tan.\theta$, therefore $\frac{ydx - xdy}{x^2} =$ $\sec.^2\theta.d\theta$, therefore $du = \frac{1}{2}x^2 \sec.^2\theta\, d\theta = \frac{1}{2}r^2 d\theta$.

12. *Let* R $=$ *the radius vector of a spiral,*
$R_1 =$ *the perpendicular on its tangent,*
$R_2 =$ *the perpendicular on the tangent of the spiral traced by* R_1,
&c. $=$ &c.; *then shall* R, R_1, R_2, R_3, &c. *decrease in geometrick progression and be inclined at equal angles.*



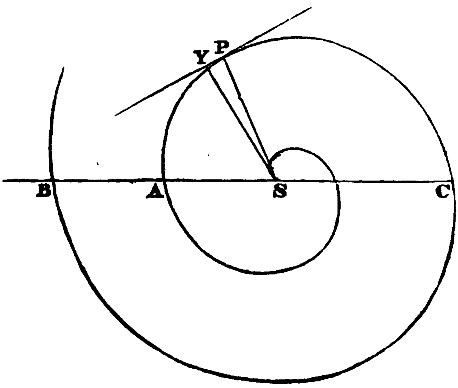
Let SA, SB, SC, &c. be the radii vectores; take Aa a small indeterminate arc; draw the tangent $ma\beta b$, cutting AB in m, SB in β, and the spiral traced by SB in b.

Since the angles at SBA, $sb\beta$ are ultimately right angles, the triangles mBβ, $sb\beta$ are similar, and consequently Bβ : $b\beta$ :: Bm : sb, which in the limit $=$ BA : SB; and ultimately the $\angle\, \beta$ is a right angle, whence SBA is similar to B$b\beta$ in the limit, and consequently similar to SBC; or SA : SB :: SB : SC and $\angle$ ASB $= \angle$ BSC.

This demonstration may be extended to any number of the spirals.

13. *Examples.*

Ex. 1. *The spiral of Archimedes.*



Def. In this spiral the radius vector is proportional to the angle which it has described.

After one complete revolution, let the radius vector have the position and magnitude of SA; then in two revolutions the radius vector as SB $=$ 2SA, &c. &c.

(1.) Required the polar equation.

$\underline{\text{SA} = a} \mid \therefore 2\pi : \theta :: a : \text{SP} = r \therefore r = \frac{a\theta}{2\pi}$ is the required equation.

(2.) The subtangent and subnormal.

$r = \frac{a}{2\pi}.\theta \therefore r' = \frac{a}{2\pi} \therefore$ subtangent which $= \frac{r^2}{r'} = \frac{2\pi r^2}{a}$.

The subnormal $= r' = \frac{a}{2\pi}$ a constant quantity.

Cor. The subtangent at A = the circumference of a circle whose radius is SA.

(3.) Tan. $\angle$ SPy = tan. $\angle$ PRQ $= \frac{\text{PQ}}{\text{QR}} = \frac{rd\theta}{dr} = \frac{r}{r'} =$ in this case $\frac{2\pi}{a}.r$; hence the angle which the radius vector makes with the curve continually increases, and when the radius vector is infinite, it approximates to a right angle as its limit.

(4.) The equation between the radius vector and the perpendicular on the tangent.

$$p = \frac{r^2}{\sqrt{r^2+r'^2}} = \frac{r^2}{\sqrt{r^2+\frac{a^2}{4\pi^2}}} = \frac{r^2}{\sqrt{r^2+b^2}}, \text{ if } b = \frac{a}{2\pi}.$$

(5.) The length.

$$ds = dr\sqrt{1+\left(\frac{r}{r'}\right)^2} = dr\sqrt{1+\left(\frac{r}{b}\right)^2} = \dots\dots$$

$$\frac{1}{b}.\,dr\,\sqrt{b^2+b^2};\ \therefore (2.\ 55\ \text{Ex. } 1.)\ \dots\dots\dots\dots$$

$$s = \frac{r}{2b}\sqrt{r^2+b^2} + \frac{b}{2}\,l\,\frac{r+\sqrt{r^2+b^2}}{b}.$$

(6.) The area.

$$du = \frac{1}{2}.r^2d\theta = \frac{r^2dr}{2b};\ \therefore u = \frac{r^3}{6b} = \frac{\pi}{3a}.r^3, \text{ for c} = 0.$$

Cor. The part included within a circle, rad. $= a, = \frac{1}{3}\pi a^2$.

(7.) This spiral does not admit of an asymptote, for when $r = \infty$, ST$' = \frac{2\pi r^2}{a} = \infty$.

(8.) The locus of T′ is a spiral whose equation is $r=\frac{a}{2\pi}\theta^2$, where θ = the angle described by ST′ + 90°.

Ex. 2. The *parabolick* spiral.

Def. In this spiral the difference between the first radius and any other is a mean proportional between the angle described and some constant quantity.

(1.) The polar equation is $r - a = b\theta^{\frac{1}{2}}$.

(2.) The equation between p and r is . . . .

$$p = \frac{2r^2(r-a)}{\sqrt{b^4 + 4r^2(r-a)^2}}.$$

(3.) The subtangent $= \frac{2r^2}{b^2}(r-a)$.

(4.) The area $= \frac{1}{b^2}\left\{\frac{r^4}{4} - \frac{ar^3}{3} + \frac{a^4}{12}\right\}$.

Ex. 3. Let $r = b.\theta^n$ be the polar equation; which includes a class of spirals.

(1.) The subtangent and subnormal.

$r' = nb\theta^{n-1}$ ∴ subtangent, which $= \frac{r^2}{r'}, = \frac{b^2\theta^{2n}}{nb\theta^{n-1}} = \frac{b}{n}\theta^{n+1}$

$= \frac{r\theta}{n}$, which shows that $ST' = \frac{1}{n}$ cir. arc PV; or $= \frac{r^{\frac{n+1}{n}}}{nb^{\frac{1}{n}}}$.

The subnormal $= r' = nb\theta^{n-1} = nb^{\frac{1}{n}}\, r^{\frac{n-1}{n}}$.

(2.) The tangent of the angle which the radius vector makes with the spiral $= \frac{r}{r'} = \frac{\theta}{n} = \frac{r^{\frac{1}{n}}}{nb^{\frac{1}{n}}}$.

(3.) Required the equation between the radius vector and the perpendicular on the tangent.

$$p = \frac{r^2}{\sqrt{r^2 + r'^2}} = \frac{r^2}{\sqrt{r^2 + n^2b^{\frac{2}{n}}\, r^{\frac{2n-2}{n}}}} = \frac{r^{\frac{n+1}{n}}}{\sqrt{r^{\frac{2}{n}} + n^2b^{\frac{2}{n}}}}.$$

If $n = 1$, the spiral is that of Archimedes or Conon's, the properties of which have been investigated.

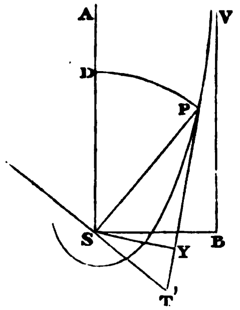
If $n = -1$, it is the *reciprocal* spiral, whose polar equation is $r\theta = b$.

Some writers call this the *hyperbolick* spiral, from the analogy which its equation bears to the equation of an hyperbola between the asymptotes.

If $n = -\frac{1}{2}$ the spiral is Cotes' Lituus (Harm. Mensurarum, *p.* 85).

(4.) When n is negative, $r = \frac{b}{\theta^n}$, and when $\theta = 0, r = \infty$, which shows that in this class of spirals the radius primus is always infinite.

(5.) When n is positive the spiral meets the pole; but when n is negative the spiral *approximates* to the pole, or approaches to it nearer than by any assignable distance.



(6.) When n is negative, the subtangent at any point of the spiral: the circular arc described by the radius vector and intercepted by the axis :: 1 : n.

For $\text{ST}' = \frac{r\theta}{n}(1) = \frac{\text{DP}}{n}$ or $\text{ST}' : \text{DP} :: 1 : n$.

(7.) When n is negative the triangle PST' : the sector PSD :: 1 : n; hence in the reciprocal spiral $\text{ST}' = \text{DP}$, and $\text{PST}' = \text{PSD}$; in the Lituus, $\text{ST}' = 2\text{DP}$ and $\text{PST}' = 2\text{PSD}$.

(8.) In the spiral of Archimedes n is positive, and the subtangent at any point P = the circular arc which would be described "*ab initio*" by the radius SP constant.

(9.) When n is negative, the spiral always admits of an asymptote, as will be seen in the following examples.

Ex. 4. The *reciprocal* spiral.

(1.) The polar equation is $r\theta = a$.

(2.) The equation between p and r is $p = \frac{ar}{\sqrt{a^2 + r^2}}$.

For $r' = -\frac{a}{\theta^2} = -\frac{r^2}{a} \therefore p = \frac{r^2}{\sqrt{r^2 + r'^2}} = \frac{ar}{\sqrt{a^2 + r^2}}$.

(3.) The subtangent $= \frac{r^2}{r'} = \frac{a^2}{\theta^2} \div -\frac{a}{\theta^2} = -a = -r\theta$; or $\text{ST}' = \text{PD} =$ a constant quantity.

ST′ is negative, which shows that ST′ and PD are on opposite sides of SP.

Cor. Hence sector SPD = Δ SPT.

(4.) The area intercepted between any two radii b and r $= \frac{1}{2} a(b-r)$.

(5.) The length.

$$ds = dr\sqrt{1 + \frac{r^2}{r'^2}} = \frac{dr}{r}\sqrt{a^2 + r^2}.$$

Hence (2. 51. Ex. 3.)

$$s = \sqrt{a^2 + r^2} + al\frac{ar}{a + \sqrt{a^2+r^2}} + \text{C}$$

$$\text{and } o = \sqrt{a^2 + b^2} + al.\frac{ab}{a + \sqrt{a^2+b^2}} + \text{C}$$

$$\therefore s = \sqrt{a^2 + r^2} - \sqrt{a^2 + b^2} + a\left\{l.\frac{ar}{a + \sqrt{a^2+r^2}} \ldots\right.$$

$$\left. - l\frac{ab}{a + \sqrt{a^2+b^2}}\right\}$$

$$= \sqrt{a^2 + r^2} - \sqrt{a^2 + b^2} + al\frac{ar + r\sqrt{a^2+b^2}}{ab + b\sqrt{a^2+r^2}}.$$

(6.) The asymptote.

Since ST′ is always $= a$, draw SB at right angles to the axis SA and = ST′; draw also BV at right angles to SB or parallel to SA, and this will be the required asymptote.

This is Cotes' 4th spiral, p. 34. See his construction for the length intercepted between two radii, p. 23.

Ex. 5. The *logarithmick* spiral.

Def. In this spiral if the angle described by the radius vector increase uniformly, the radii increase or decrease in a geometrick progression.

(1.) The polar equation is $r = a^\theta$ where a is the base of the system of logarithms to which r and θ belong.

(2.) The equation between p and r is $p = \dfrac{r}{\sqrt{1 + \text{A}^2}}$ where $\text{A} = la$; for $r' = \text{A}r$.

Cor. Hence the radius vector makes always the same angle with the curve; for $\dfrac{p}{r}$ is constant.

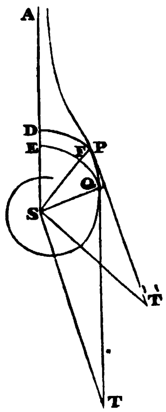
It is in consequence sometimes called the *equiangular* spiral.

(3.) The subtangent $= \frac{r^2}{r'} = \frac{r}{\text{A}} = \text{M} \times r$, if M is the modulus of the system.

Cor. 1. The tangent of the $\angle$ which the radius makes with the curve $= \frac{\text{ST}'}{\text{SP}} = \text{M}$ the modulus of the system; hence, by varying the spiral, we may construct by means of it the logarithms of any proposed system. If, for instance, it were required to find the spiral in which the angles are the *hyperbolick* logarithms of the radii which intercept them, we have $\text{M} = 1$, or tan. $\angle$ SPY $= 1$, or SPY $= 45°$.

Cor. 2. The circle is an equiangular but not a logarithmick spiral; for when tan. $\angle$ SPY $= 90°$, $\text{M} = \infty$.

(4.) The area intercepted between two radii b and $r = \frac{\text{M}}{4}(r^2 - b^2)$; and the whole area to the centre $= \frac{\text{M}r^2}{4} = \frac{1}{2} \Delta$ SPT$'$ and is therefore finite.



Ex. 6. The *lituus*.

(1.) The polar equation is $r = a\theta^{-\frac{1}{2}}$.

(2.) The equation between p and r is $p = \frac{b^2 r}{\sqrt{b^4 + r^4}}$, where . . . $b^2 = 2a^2$.

(3.) The subtangent $= -\frac{b^2}{r}$.

Hence, take SA the axis which is infinite, and constructing the figure as before, $\text{ST}' = \frac{b^2}{r} = \frac{2a^2}{\text{SP}}$

$\therefore \frac{\text{ST}' \times \text{SP}}{2} = a^2 = r^2\theta = 2 \times \frac{r.r\theta}{2}$;

which shows that Δ SPT$'$ = twice sector SPD = a constant quantity.

Since 2SPD = SPT$'$ $\therefore$ DP $= \frac{1}{2}\text{ST}' = \frac{a^2}{\text{SP}}$, which leads to this

construction; take SD any variable distance in the axis; draw the circular arc DP, and take it $= \frac{a^2}{\text{SD}}$, and P traces the spiral.

(4.) The area intercepted between two radii SP and SQ $= a^2 . l \frac{\text{SP}}{\text{SQ}} = \triangle \text{SPT}' \times l \frac{\text{SP}}{\text{SQ}}$.

Cor. Describe the circular arc QFE cutting SP in F; then sector SEQ = SDP ∴ SFQ = DEFP; add to each PFQ, then SPQ = DEQP; hence DEQP $= \triangle \text{SPT}' \times l \frac{\text{SD}}{\text{SE}}$; or in Cotes' language, DEQP is the measure of the ratio of SD : SE or SQ to the modulus, the $\triangle$ SPT′.

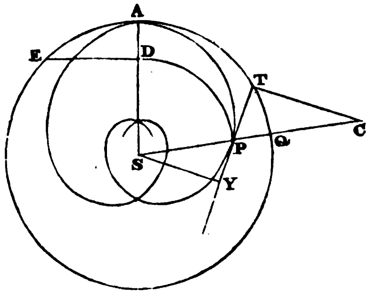
If SE be taken the first abscissa, and SD increase in geometrick, the area DEQP will increase in arithmetick progression.

(5.) The axis is the asymptote.

(6.) Required the point where the tangent is parallel to the axis.

Let Q be the required point, and draw its subtangent ST; then since ∠ SQT = ∠ QSE, and that ST = 2QE, the angle must be such that the tangent = twice its arc, or ∠ = 66° +.

Cotes remarks that the hyperbola between the asymptotes bears the same kind of relation to this spiral that the Apollonian parabola bears to the spiral of Archimedes, or the logarithmick curve (vid. Ch. 13.) bears to the reciprocal spiral.



Ex. 7. The *hyperbolick*, or Cotes' first spiral.

The equation between p and r is $p = \frac{br}{\sqrt{a^2 - r^2}}$.

(1.) The polar equation.

$$\frac{br}{\sqrt{a^2 - r^2}} = p = \frac{r^2}{\sqrt{r^2 + r'^2}}$$

$$\therefore r'^2 = \frac{(a^2 - b^2)r^2 - r^4}{b^2}.$$

Substitute $c^2 = a^2 - b^2$; then $\frac{dr}{d\theta} = r' = \pm \frac{r\sqrt{c^2 - r^2}}{b}$, from

which it appears that r cannot be greater than c. Describe a circle with radius SA $=c$, and take SA as the radius primus; then

if ASP $= \theta$, $d\theta = \dfrac{-bdr}{r\sqrt{c^2-r^2}}$ $\therefore \theta = \dfrac{b}{c}\, l. \dfrac{c+\sqrt{c^2-r^2}}{r}$, which

is the required equation where $c = \sqrt{a^2-b^2}$.

Cor. 1. When $r = 0$, $\theta = \infty$, or the radius vector makes an infinite number of revolutions before it vanishes.

Cor. 2. To construct the spiral.

Take SD = the variable distance; draw DE the ordinate of the circle; join SE *, and always take AQ the measure of the ratio SE + ED : SD to the modulus $b = \sqrt{a^2-c^2}$; join SQ, and in it take SP = SD, and P traces the spiral.

Note.—This spiral has only *two* arbitrary constants.

(2.) The subtangent $= \dfrac{br}{\sqrt{c^2-r^2}}$; and the tangent . . . .

$$= \frac{r\sqrt{a^2-r^2}}{\sqrt{c^2-r^2}}.$$

(3.) The area.

$$du = \tfrac{1}{2}r^2d\theta = -\tfrac{1}{2}\frac{brdr}{\sqrt{c^2-r^2}} \therefore u = \frac{b}{2}\sqrt{c^2-r^2} \propto \text{DE}.$$

The area to the centre $= \dfrac{bc}{2}$.

Ex. 8. Cotes' *third spiral.*

The equation between p and r is $p = \dfrac{br}{\sqrt{a^2+r^2}}$ where a is greater than b.

If $b = a$, the curve is the reciprocal spiral, Ex. 4.

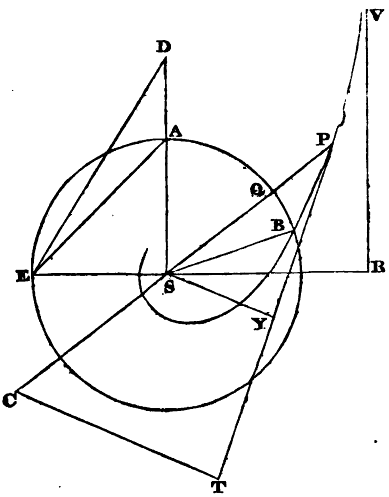
If b is greater than a, the equation belongs to the elliptick spiral.

(1.) The polar equation.

$\dfrac{b^2}{a^2+r^2} = \dfrac{r^2}{r^2+r'^2}$ $\therefore br' = r\sqrt{r^2+c^2}$ where $c^2 = a^2-b^2$, or

* SE is omitted in the diagram.

$r' = \frac{r}{b}\sqrt{r^2+c^2}$



where there is no limit either to the increase or the decrease of r; describe then a circle with radius $= c$, and let it cut the spiral in B; and first suppose BSP $= \theta$, SP increasing as θ increases, consequently

$$d\theta = \frac{bdr}{r\sqrt{r^2+c^2}},$$

wherefore

$$\left.\begin{array}{l} \text{BQ} = b\,l.\dfrac{\sqrt{r^2+c^2}-c}{r} + \text{C} \\ 0 = b\,.\,l(\sqrt{2} - 1) + \text{C} \end{array}\right\} \therefore \text{BQ} = b\,l\frac{\sqrt{r^2+c^2}-c}{r(\sqrt{2}-1)};$$

hence when $r = \infty$, BQ $= b.\,l\dfrac{1}{\sqrt{2}-1}, = bl(\sqrt{2}+1), \ldots$

which is a finite positive quantity; take therefore . . . . BA $= b\,.\,l(\sqrt{2}+1)$, and considering SA as the radius primus, we shall have . . . .

$$d\theta = \frac{-bdr}{r\sqrt{r^2+c^2}} \text{ and AQ} = b\,l\frac{\sqrt{r^2+c^2}+c}{r} \text{ for the polar}$$

equation.

Cor. Hence to construct the spiral. Let SAD be the radius primus; in it take SD the variable distance; describe a circle EAB whose radius $= \sqrt{a^2 - b^2}$; draw the radius SE perpendicular to SA; join DE, and always take AQ the measure of the ratio of DE + SE : SD to the modulus b; join SQ, and in it produced, if necessary, take SP = SD, and P will trace the spiral.

(2.) The subtangent $= \dfrac{br}{\sqrt{r^2+c^2}}$.

(3.) The area BSP $= \frac{b}{2} . (\text{DE} - \text{EA})$; and the area to the centre $= \frac{b}{2}(\text{DE} - \text{SE})$.

(4.) The asymptote.

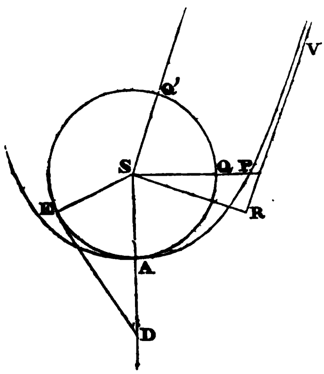
When $r = \infty$, the subtangent $= b$; draw SR perpendicular to SA and $= b$, and a line RV perpendicular to SR is the required asymptote.

Ex. 9. The *elliptick*, or Cotes' fifth spiral.

The equation is $p = \frac{br}{\sqrt{a^2 + r^2}}$ where a is less than b.

(1.) The polar equation.

$\frac{b^2}{a^2 + r^2} = \frac{r^2}{r^2 + r'^2} \therefore r' = \frac{r}{b}\sqrt{r^2 - c^2}$ where $c^2 = b^2 - a^2$, and consequently r cannot be less than c.



Describe then a circle with radius SA $= c \ldots . = \sqrt{b^2 - a^2}$, and taking SA as the radius primus, we have $d\theta = \frac{bdr}{r\sqrt{r^2 - c^2}}$,

$\therefore \angle \text{ASP} = \frac{b}{c}\sec.^{-1}\frac{r}{c}$; or

$\cos. \frac{c\theta}{b} = \frac{c}{r}$ is the required equation.

Cor. Hence, Cotes' construction. "In SA produced take the variable distance SD; draw DE a tangent to the circle whose radius = SA; take AQ : AE : : $b : c$; join SQ, and produce it making SP = SD, and P is a point in the spiral."

(2.) The subtangent $= \frac{br}{\sqrt{r^2 - c^2}}$.

(3.) The area SAP $= \frac{b}{2} . \text{DE} \propto \text{DE}$.

(4.) The asymptote.

When $r = \infty$, the subtangent $= b$, and AE $= a$ qua-

drantal arc; wherefore take AQ′ : a quadrant :: b : c; join SQ′; draw SR perpendicular to SQ′, and $= b$; and through R draw RV perpendicular to SR.

Ex. 10. The *centrifugal* spiral.

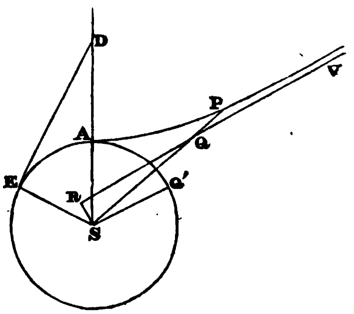
The equation is $p = \dfrac{br}{\sqrt{r^2-a^2}}$.

(1.) The polar equation.

$\dfrac{b^2}{r^2-a^2} = \dfrac{r^2}{r^2+r'^2}$ $\therefore$ $br' = r\sqrt{r^2-c^2}$ where $c^2 = a^2+b^2$,

$\therefore$ $d\theta = \dfrac{bdr}{r\sqrt{r^2-c^2}}$ $\therefore$ $\angle$ ASP $= \dfrac{b}{c}\sec.^{-1}\dfrac{r}{c}$.

Hence the following construction: in SA produced take any distance SD; draw DE the tangent to the circle, and take AQ : AE :: b : c; produce SQ to P, making SP $=$ SD; and P is a point in the spiral.



(2.) The subtangent $= \dfrac{br}{\sqrt{r^2-c^2}}$.

(3.) The area ASP $= \dfrac{b}{2}$ DE as in the elliptick spiral.

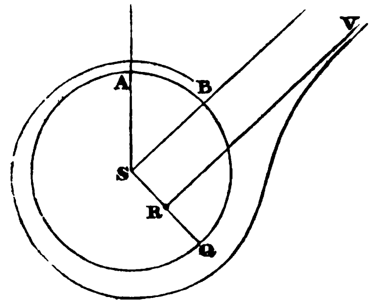
(4.) The asymptote.

When $r = \infty$, the subtangent $= b$, and AE $=$ a quadrant; wherefore take AQ′ $= \dfrac{b}{c} \times$ quadrant; join SQ′; draw SR $= b$ and perpendicular to SQ′; and a line drawn through R parallel to SQ′ will be the asymptote.

Ex. 11. A spiral whose polar equation is $r = \dfrac{a\theta^2}{\theta^2-1}$.

$\theta^2 = \dfrac{r}{r-a}$; when $r = \infty$, $\theta = 1$; therefore $\theta = 57^\circ\ 17'\ 44''.981$ (2. 18.).

Let SA be the position of the radius primus; take SA $= a$, and describe the circle ABQ with radius SA; no part of the spiral can be within this circle; and since when $r = a$, $\theta = \infty$, the circle is an asymptote to the spiral. Also make $\angle$ ASB $= 57^\circ\ 17'\ 44''.981$; then SB is the ultimate direction of the infinite branch.



(1.) Required the equation between r and p.

$$\theta = \left(\frac{r}{r-a}\right)^{\frac{1}{2}} \therefore d\theta = \frac{1}{2}\left(\frac{r-a}{r}\right)^{\frac{1}{2}}\left\{\frac{dr}{r-a} - \frac{rdr}{(r-a)^2}\right\} \ldots$$

$$= \frac{-adr}{2r^{\frac{1}{2}}(r-a)^{\frac{3}{2}}} \therefore r' = \frac{-2r^{\frac{1}{2}}(r-a)^{\frac{3}{2}}}{a} \therefore p = \frac{ar^2}{\sqrt{a^2r^2+4r(r-a)^3}}$$

$$= \frac{ar^{\frac{3}{2}}}{\sqrt{a^2r+4(r-a)^3}}.$$

(2.) Required to draw the rectilinear asymptote.

When $r = \infty$, $p = \dfrac{ar^{\frac{3}{2}}}{2r^{\frac{3}{2}}} = \dfrac{a}{2}$; hence draw the radius SQ perpendicular to SB; bisect SQ in R; draw RV perpendicular to SQ, or parallel to SB and RV is an asymptote.

(3.) If SB be taken the radius primus, the equation is

$$\frac{r}{r-a} = (\theta + 1)^2.$$

(4.) There is no algebraical relation between the co-ordinates of the axis (Art. 3. Cor. 3.)

14. PRAXIS.

1. The radius vector of a circle is produced till the part

produced is equal to the sine of the angle described; required the area of the spiral.

The equation is $\sin.\theta = \frac{r-a}{a}$: it intersects the axis at $\angle\ 45°$; and $(u) = a^2 + \frac{3\pi a^2}{8}$.

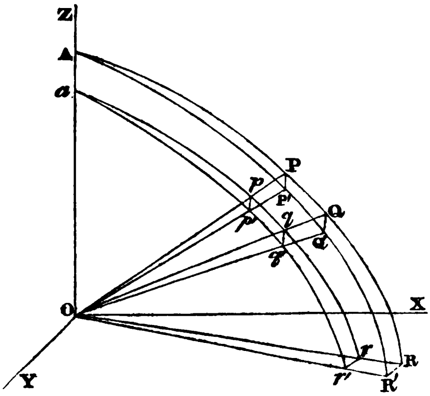
2. Two points set out together at the same time from A and B the extremities of the diameter of a circle, and moving in the same direction, arrive at the same time at A; required the area of the spiral traced by the point which bisects their distance.

The equation is $\frac{r}{a} = \cos.\frac{1}{3}(2\pi - \theta)$. The whole area included within the spiral $= \frac{3}{4}$ of the circle. The area of the foliate $= \frac{a^2}{4}\left(\pi - \frac{3\sqrt{3}}{2}\right)$; and the equation of the co-ordinates is $4(x^2 + y^2)^2 - 3a^2(x^2 + y^2) + a^3x = 0$.

The rectification of this curve depends upon that of an ellipse.

15. *Solids.* By transferring the polar co-ordinates to the rectangular, the solid formed by the revolution of part of the spiral round the axis may be calculated: this calculation is seldom required. We shall give in the following article a formula for finding the content of a co-ordinate solid by considering it as a spiral, or by referring all the points of which it is composed to a fixed pole and axis.

16. *Required the content of a solid considered as a spiral.*



Let P be any position of the generating point within the solid; with OP as radius, describe in the plane ZOP the quadrant APR; take PQ a small increment of AP; let APR revolve through the small spherical angle RAR', P, Q and R describing the circular arcs PP', QQ', and RR'.

Also, in the planes AOR, AOR$'$ draw the quadrants apr, $ap'r'$ near to APR, AP$'$R$'$ cutting OP and OP$'$ in p and p', and ρQ and OQ$'$ in q and q'.

Whatever motion be assigned to P, by assuming PQ, PP$'$ and Pp in the requisite ratio, they may be made, in the limit, to represent its motion in three directions, each of which is at right angles to the other two. Hence the total increment of the solid arising from the three partial increments in these directions is a curvilinear prism whose base is the surface PP$'$Q$'$Q, and altitude Pp; and since all the angles of this prism are right angles, its solid content, in the limit $=$ PQ $\times$ PP$'$ $\times$ Pp.

Let $\angle$ ZOP $= \theta$	then PQ $= rd\theta$, PP$'$ $= \sin.\theta.$RR$' = sin.\theta.rd\varphi$
$\angle$ XOR $= \varphi$	and Pp $= dr$;
OP $= r$	wherefore $u = \int\int\int d\theta d\varphi dr . r^2 \sin.\theta$.

Cor. If the solid is a solid of revolution, $u = \ldots\ldots$ $2\pi\int\int d\theta dr . r^2 \sin.\theta$.

For in this case φ is an independent function; and $d\varphi$ is to be integrated between $\varphi = 0$, $\varphi = 2\pi$.

Ex. A sphere.

$u = 2\pi\int r^2 dr \int d\theta \sin.\theta$; but $\int d\theta \sin.\theta = -\cos.\theta$, and this is to be integrated between $\theta = 0$, $\theta \doteq 2\pi$; $\therefore (\int d\theta \sin.\theta) = 2$

$\therefore u = 4\pi\int r^2 dr = \dfrac{4\pi r^3}{3}$; for the origin of the fluent is $r = 0$.

CHAPTER XI.

Curvature; Involutes and Evolutes; Osculating Curves.

1. *Definitions.*

(1.) The *radius of curvature* at any point of a curve is the radius of that circle which touches the curve and has the same curvature at that point.

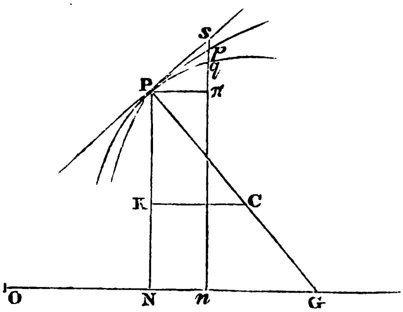
(2.) The *sub-radius of curvature* is a line drawn from the centre of the circle of curvature perpendicular on the ordinate.

(3.) The *co-radius* is that part of the ordinate which is intercepted between the curve and the sub-radius.

Thus, if C is the centre of the circle of curvature at P, and CK be drawn perpendicular on PN, PC is the radius, CK the sub-radius, and PK the co-radius.

The sub-radius and the co-radius are the semi-chords of curvature parallel and perpendicular to the axis.

2. *If an ordinate* nps *be drawn near to* NP, *cutting the curve in* p, *and the tangent at* P *in* s, *the limit of* $\frac{sp}{Nn^2}$ *shall* $= \mp \frac{\frac{1}{2}d^2y}{dx^2}$, *where* x *and* y *are the co-ordinates of* P.



For drawing $P\pi$ perpendicular on ns, by Taylor's Theorem,

$$\pi p = \frac{dy}{dx}\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3} + \&c.$$

and (8. 2.) $\pi s = \frac{dy}{dx}\cdot\frac{h}{1}$,

therefore $sp = \mp \frac{d^2y}{dx^2}\,\frac{h^2}{1.2} \mp \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3} \mp$ &c. according as the curve is concave to the axis or convex; whence $\frac{sp}{h^2} = \mp\frac{d^2y}{dx^2}\cdot\frac{1}{1.2}\mp\frac{d^3y}{dx^3}\,\frac{h}{1.2.3}\mp$, &c. which in the limit $=\mp\frac{\frac{1}{2}d^2y}{dx^2}$.

3. *If the circle of curvature be drawn at the point* P *cutting* np *in* q, *the limit of* $\frac{\mathrm{sq}}{\mathrm{Nn}^2}$ *shall* $= \frac{\mp\frac{1}{2}\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2}$.

For (7. 34. Cor.) the perpendicular subtenses of Pp and Pq are equal, and consequently the subtenses inclined at any the same angle are equal; or $sq = sp$ in the limit; and the limit of $\frac{sq}{\mathrm{N}n^2} = \frac{\mp\frac{1}{2}d^2y}{dx^2}$.

4. *Required to find the magnitude and position of the circle of curvature at any point of a known curve.*

Let $\frac{dy}{dx} = p$, $\frac{d^2y}{dx^2} = q$; then these quantities are the same in the curve at P, and in its circle of curvature; draw the normal PCG; C, the centre of curvature, is in PG; draw its co-ordinates OM, MC *.

ON $= x$	The circle's equation is $(\alpha-x)^2+(y-\beta)^2=\mathrm{R}^2$,
NP $= y$	from which, in order to determine R, we must
CP $=$ R	eliminate α and β; whence, differentiating twice,
OM $= \alpha$	there results $-(\alpha - x) + (y - \beta)p = 0$, and
MC $= \beta$	$1 + p^2 + (y - \beta)q = 0$; or . . . .

$y - \beta = \frac{1+p^2}{-q}$ (1), and $\alpha - x = \frac{p}{-q}(1 + p^2)$ (2).

Hence, by substitution, $\mathrm{R}^2 = \frac{(1+p^2)^2}{q^2} + \frac{p^2(1+p^2)^2}{q^2}$ . . .

$= \frac{(1+p^2)^3}{q^2}$ and $\mathrm{R} = \frac{(1+p^2)^{\frac{3}{2}}}{-q} = \frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}}$; which is the radius in terms of the fluxional coefficients of the curve's equation.

And to find the *position* of the circle; we have

* OM, MC are omitted in the diagram.

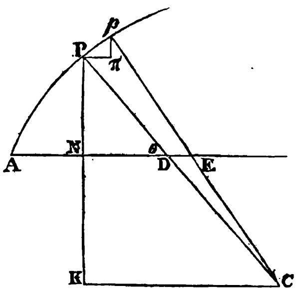
$\alpha = x - \frac{p}{q}(1 + p^2)$ and $\beta = y + \frac{1+p^2}{q}$; which are the co-ordinates of its centre.

This value of R will be positive when the curve is concave; for in that case $\frac{d^2y}{dx^2}$ is negative. If the value of R then as calculated from the equation proves to be negative, the curve is convex and the radius must be drawn *from* the axis.

5. Another method of investigating the radius of curvature. (Encyclopædia Britan. Art. Fluxions.)

AN $= x$	
NP $= y$	
AP $=$ S	
CP $=$ R	
$\angle$ ADP $= \theta$	

Take Pp a small indeterminate arc; draw Pπ parallel and $p\pi$ perpendicular to the axis. Draw also the normals PC, pC; then the limit of C is the centre of the circle of curvature: let CP, Cp cut the axis in D and E.



Then, inc.$\theta = \angle$ AEp $- \angle$ ADP $= \angle$ PCp $= \frac{\text{P}p}{\text{CP}}$ in the limit; therefore limit of $\frac{\text{inc.}\theta}{\text{inc.AP}} = \frac{1}{\text{CP}}$; and since C may be supposed fixed when Pp is taken in the limit, we have $\frac{d\theta}{ds} = \frac{1}{\text{CP}} = \frac{1}{\text{R}}$, and $d\theta = \frac{ds}{\text{R}}$.

But $\tan.\theta = \frac{\text{PN}}{\text{ND}} = \frac{\text{P}\pi}{p\pi}$ in the limit $= \frac{dx}{dy}$: therefore . . . $d.\tan.\theta = \frac{d^2x}{dy} - \frac{dxd^2y}{dy^2} = \frac{d^2xdy - dxd^2y}{dy^2}$; whence $d\theta$, which $= \frac{d.\tan.\theta}{1+\tan.^2\theta} = \frac{d^2xdy - d^2ydx}{dy^2} \div \left(1 + \frac{dx^2}{dy^2}\right) = \frac{d^2xdy - d^2ydx}{ds^2}$; and consequently $\frac{ds}{\text{R}} = \frac{d^2xdy - d^2ydx}{ds^2}$ and $\text{R} = \ldots\ldots\ldots \frac{ds^3}{d^2xdy - d^2ydx}$.

Cor. 1. If dx be constant, $d^2x = 0$ and $\text{R} = \frac{ds^3}{-d^2ydx} \dots$
$= \frac{(1+p^2)^{\frac{3}{2}}}{-q}$.

Cor. 2. If dy be constant, $\text{R} = \frac{ds^3}{d^2xdy}$.

Cor. 3. If ds be constant, $\text{R} = \frac{dsdy}{d^2x}$.

For $ds^2 = dx^2 + dy^2$, therefore, differentiating, $0 = dxd^2x + dyd^2y$; or $d^2y = \frac{-dxd^2x}{dy}$, and $\text{R} = \frac{ds^3}{d^2xdy + \frac{dx^2d^2x}{dy}} \dots$
$= \frac{dsdy}{d^2x}$.

Cor. 4. On the same supposition, $\text{R} = -\frac{dsdx}{d^2y}$.

Cor. 5. On the same supposition, $\text{R} = \frac{ds^2}{\sqrt{d^2y^2 + d^2x^2}}$.

For $\text{R}^2 = \frac{ds^6}{(dyd^2x - dxd^2y)^2}$; but differentiating $ds^2 = dx^2 + dy^2$, $0 = dxd^2x + dyd^2y$ . . . .

$\therefore \text{R}^2 = \frac{ds^6}{(dyd^2x - dxd^2y)^2 + (dxd^2x + dyd^2y)^2}$ . . . .

$= \frac{ds^6}{(dx^2+dy^2)(d^2x^2+d^2y^2)} = \frac{ds^4}{d^2x^2+d^2y^2} \therefore \text{R} = \frac{ds^2}{\sqrt{d^2x^2+d^2y^2}}$.

Cor. 6. The co-radius $= \frac{1+p^2}{-q}$.

Cor. 7. The sub-radius $= \frac{p(1+p^2)}{-q}$.

6. *Examples.*

Ex. 1. The Apollonian parabola.

$y^2 = ax \therefore y = a^{\frac{1}{2}}x^{\frac{1}{2}} \therefore p = \frac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{2}} \therefore q = -\frac{1}{4}a^{\frac{1}{2}}x^{-\frac{3}{2}} \therefore$

$$\text{R} = \frac{\left(1 + \frac{a}{4x}\right)^{\frac{3}{2}}}{\frac{1}{4}a^{\frac{1}{2}}x^{-\frac{3}{2}}} = \frac{(a+4x)^{\frac{3}{2}}}{2a^{\frac{1}{2}}}.$$

When $x = 0$, $2\text{R} = a$, which therefore is the diameter of curvature at the vertex.

Ex. 2. Let the equation of the parabola be $a^{n-1}x = y^n$;

$$\therefore p = \frac{a^{n-1}}{ny^{n-1}} \therefore q = \frac{dp}{dx} = \frac{-(n-1)}{n}\frac{a^{n-1}p}{y^n} = \frac{-(n-1)}{n^2}\cdot\frac{a^{2n-2}}{y^{2n-1}}$$

$$\therefore \text{R} = \frac{\left(1 + \dfrac{a^{2n-2}}{n^2y^{2n-2}}\right)^{\frac{3}{2}}}{\dfrac{n-1}{n^2}\cdot\dfrac{a^{2n-2}}{y^{2n-1}}} = \frac{(n^2y^{2n-2} + a^{2n-2})^{\frac{3}{2}}}{n(n-1)a^{2n-2}y^{n-2}}.$$

$$\text{If } n = 2,\ \text{R} = \frac{(4y^2 + a^2)^{\frac{3}{2}}}{2a^2} = \frac{(4x + a)^{\frac{3}{2}}}{2a^{\frac{1}{2}}}.$$

$$\text{If } n = 3,\ \text{R} = \frac{(9y^4 + a^4)^{\frac{3}{2}}}{6a^4y}.$$

At the vertex $y = 0$, and R is finite only when $n = 2$: if n be less than 2, $\text{R} = 0$, or the curvature is infinite; if n be greater than 2, the curvature vanishes.

$$\text{The subradius} = \frac{1}{n-1}\left\{\text{subtangent} + \text{subnormal}\right\}.$$

Cor. 1. If the curve be an hyperbola between the asymptote whose equation is $y^nx = a^{n+1}$.

$$\text{R} = -\frac{(n^2a^{2n+2} + y^{2n+2})^{\frac{3}{2}}}{n(n+1)a^{n+1}y^{2n+1}}.$$

Cor. 2. In the Apollonian parabola, the subradius = the sum, and in the rectangular hyperbola, the semi-sum of the subtangent and subnormal.

Ex. 3. An ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \therefore p = \frac{-b^2x}{a^2y} \therefore -q = \frac{-dp}{dx} = \frac{b^2}{a^2y} \ldots$$

$$-\frac{b^2x}{a^2}\cdot\frac{p}{y^2} = \frac{b^2}{a^2y} + \frac{b^4x^2}{a^4y^3} \therefore \text{R} = \frac{\left(1 + \dfrac{b^4x^2}{a^4y^2}\right)^{\frac{3}{2}}}{\dfrac{b^2}{a^2y} + \dfrac{b^4x^2}{a^4y^3}} \ldots\ldots$$

$$=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^2y^2+a^2b^4x^2}=\frac{a^6b^6\left(\frac{y^2}{b^4}+\frac{x^2}{a^4}\right)^{\frac{3}{2}}}{a^4b^4\left(\frac{y^2}{b^2}+\frac{x^2}{a^2}\right)}=a^2b^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)^{\frac{3}{2}}.$$

Cor. At the vertex $2\text{R}=\frac{2b^2}{a}$ = the latus rectum of the ellipse.

Ex. 4. $ax-ay=x^2+y^2$.

$$p=\frac{a-2x}{a+2y}\therefore q=\frac{dp}{dx}=\frac{-2}{a+2y}-\frac{2(a-2x)p}{(a+2y)^2}\ \ldots\ldots\ldots$$

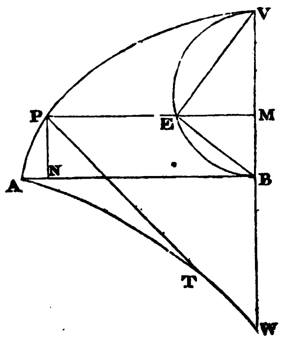
$$=\frac{-2}{a+2y}\left\{1+\left(\frac{a-2x}{a+2y}\right)^2\right\}\therefore\text{R}=\frac{\left\{1+\left(\frac{a-2x}{a+2y}\right)^2\right\}^{\frac{3}{2}}}{\frac{2}{a+2y}\left\{1+\left(\frac{a-2x}{a+2y}\right)^2\right\}}$$

$$=\frac{a+2y}{2}\left\{1+\left(\frac{a-2x}{a+2y}\right)^2\right\}^{\frac{1}{2}}=\tfrac{1}{2}\left\{(a+2y)^2+(a-2x)^2\right\}^{\frac{1}{2}}$$

$$=\tfrac{1}{2}\left\{2a^2+4ay+4y^2-4ax+4x^2\right\}^{\frac{1}{2}}=\frac{a}{\sqrt{2}}.$$

This is the equation of a quadrant whose chord $=a$.

Ex. 5. The cycloid.



(5. Cor. 3) $\text{R}=\frac{dsdy}{d^2x}$ when ds is constant.

But by a property of the cycloid VP^2 : VA^2 : : VM : VB or $s^2 : 4a^2 : : a-y : a \therefore a-y=\frac{s^2}{4a} \because dy=-\frac{sds}{2a}$.

Also $dx^2=ds^2-dy^2=ds^2-\frac{s^2ds^2}{4a^2}\therefore dx=\frac{(4a^2-s^2)^{\frac{1}{2}}ds}{2a}$

$\therefore$ differentiating, $d^2x=\frac{(4a^2-s^2)^{-\frac{1}{2}}sds^2}{2a}\therefore\text{R}=\frac{s}{2a}\ \ldots\ldots$

$$\div \frac{(4a^2-s^2)^{-\frac{1}{2}}s}{2a} = \sqrt{4a^2-s^2} = \sqrt{4\text{VB}^2 - 4\text{VE}^2} = 2\text{BE}.$$

Cor. The co-radius = 2BM and the sub-radius = 2EM. (Vince's Fluxions).

In article 5 it was assumed that the limit of the intersections of PC and pC is the centre of the circle of curvature. This principle, which is frequently referred to in investigating the properties of spirals, may be demonstrated by a direct analytical investigation.

7. *Required to find the point which is the limit of the intersections of two consecutive normals.*

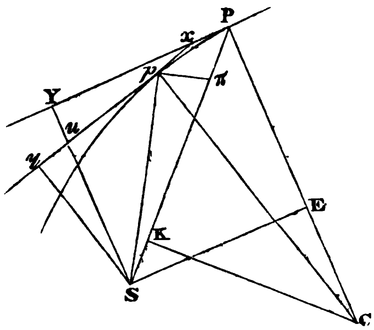
Let PC and pC be the normals;

The equation of PC is $x - x' = -p(y - y')$, (8. 4) and the characteristick property of C is that the *initial* motion of P does not produce any change in its position; if then x', y' are its co-ordinates, and the equation be differentiated on the supposition that x' and y' are constant, the resulting values of x', y' belong to C.

Hence, we have $1 = -q(y-y') - p^2$, or $y' = y + \dfrac{1+p^2}{q}$;

and consequently $x' = x - \dfrac{p}{q}(1+p^2)$ which are the co-ordinates of the centre of the circle of curvature.

8. *Required the radius of curvature of a spiral, and the chord passing through its pole.*



Take Pp a small indeterminate arc, draw from the pole S the perpendiculars SY and Sy on the tangents at P and p; produce yp to meet PY in x; draw also the normals PC, pC which ultimately meet in the centre of the circle of curvature.

Draw $p\pi$, CK perpendicular on SP.

SP = r	Then CP and Cp are respectively parallel to SY and Sy; hence, taking all the ratios in their limit, we have
SY = p	
CP = R	

$$\begin{array}{llll} \text{Y}u : yu :: & \text{Y}x : sy :: & \text{PY} : \text{SY} \\ yu : \text{P}p :: & & \text{SY} : \text{CP} \\ \text{P}p : p\pi :: & & \text{SP} : \text{PY} \end{array}$$

therefore $dp : \pm dr ::$ $r : \text{R}$ which therefore $= \frac{rdr}{dp}$.

*Otherwise.** Join SC, and draw SE perpendicular on CP, then (Eu. 3. 12.) $\text{SC}^2 = \text{SP}^2 + \text{CP}^2 - 2\text{CP}\,.\,\text{PE}$; and differentiating on the supposition that C is fixed, we have $0 = 2rdr - 2\text{R}dp$, or $\text{R} = \frac{rdr}{dp}$. (Vince's Fluxions).

Also the chord $= 2\text{PK} =$ from similar triangles, $\frac{2\text{PC}.\text{SY}}{\text{SP}}$ $= \frac{2pdr}{dp}$.

9. *Def.* PK is the *co-radius*; and CK the *sub-radius* of curvature.

10. *Required the radius of curvature in terms of the polar co-ordinates.*

(10. 7) $p = \frac{r^2}{(r^2+r'^2)^{\frac{1}{2}}}$, therefore $dp = \frac{2rdr}{\sqrt{r^2+r'^2}}$ ......

$-\frac{r^2(rdr+drr'')}{(r^2+r'^2)^{\frac{3}{2}}} = \frac{r^3dr+2rr'^2dr-r^2drr''}{(r^2+r'^2)^{\frac{3}{2}}}$ if $d\theta = 1$; therefore

$\frac{dp}{rdr} = \frac{r^2+2r'^2-rr''}{(r^2+r'^2)^{\frac{3}{2}}}$; whence R, which $= \frac{rdr}{dp}$, $= \frac{(r^2+r'^2)^{\frac{3}{2}}}{r^2+2r'^2-rr''}$,

or $= \frac{(r^2+p^2)^{\frac{3}{2}}}{r^2+2p^2-rq}$, if $p = \frac{dr}{d\theta}$ and $q = \frac{d^2r}{d\theta^2}$.

Cor. The sub-radius CK, if the fluxional triangle PQR be drawn, $= \frac{\text{CP}\times\text{QR}}{\text{PR}} = \frac{(r^2+p^2)^{\frac{3}{2}}}{r^2+2p^2-rq} \times \frac{p}{\sqrt{r^2+p^2}} = \frac{p(r^2+p^2)}{r^2+2p^2-rq}$.

The co-radius $\text{PK} = \frac{\text{CP}\times\text{PQ}}{\text{PR}} = \frac{(r^2+p^2)^{\frac{3}{2}}}{r^2+2p^2-rq} \times \frac{r}{\sqrt{r^2+p^2}}$

---

* SC is omitted in the diagram.

$= \frac{r(r^2+p^2)}{r^2+2p^2-rq}$, or the chord of curvature passing through the pole, which $= 2\text{PK}$, $= \frac{2r(r^2+p^2)}{r^2+2p^2-rq}$.

11. *Def.* The *evolute* of a curve is the curve which is traced by the centres of its circles of curvature; the original curve being the *involute.*

12. *The radius of curvature of the involute is a tangent to the evolute.*

The *initial* variation of the co-ordinates of any point of the involute produces no change either in the magnitude of the radius of curvature, or in the co-ordinates of the corresponding point of the evolute (Art. 7.); but if the co-ordinates of the evolute vary, the co-ordinates of the corresponding point of the involute and the radius of curvature are necessarily changed.

Let v & w be the co-ordinates of any point of the evolute,
x & y those of the corresponding point of the involute,
R = the radius of curvature of the involute.

Then, since the radius of curvature is a normal line to the involute, we have $(y-w) = -\frac{1}{p}(x-v)$ (8. 4.).

Now, from the equation of the circle, $(x-v)^2+(y-w)^2 = \text{R}^2$; and differentiating twice on the supposition that x and y alone vary, we have $x-v+(y-w)p = 0$ (1), and $1+(y-w)q+p^2 = 0$ (2).

Next, differentiate (1) on the supposition that all the magnitudes vary; then we have, since they are all functions of x, $1-\frac{dv}{dx}+(y-w)q+p^2-\frac{dw}{dx}p = 0$; from which subtract (2), and there results $-\frac{dv}{dx}-\frac{dw}{dx}p$ (3) $= 0$, or $p = -\frac{dv}{dx} \div \frac{dw}{dx} =$ (4. 14.) $-\frac{dv}{dx}\frac{dx}{dw} =$ (3. 8.) $-\frac{dv}{dw}$; which, substituted in the equation of the radius of curvature, gives $y-w = \frac{dw}{dv}(x-v)$, which is the equation of the evolute's tangent (8. 3.).

13. *The arc of the evolute is equal to the radius of curvature of the involute $\pm$ a constant quantity.*

$(x-v)^2 + (y-w)^2 = \text{R}^2$; whence, differentiating on the supposition that all the magnitudes vary, $(x-v)\left\{1 - \frac{dv}{dx}\right\}$ $+ (y-w)\left\{p - \frac{dw}{dx}\right\} = \frac{\text{R}d\text{R}}{dx}$.

But by preceding Art., equations (1) and (2), $x - v + p.(y-w) = 0$, and $1 + p^2 + (y-w)\,q = 0$; whence $y - w$ $= -\frac{1+p^2}{q}$, and $x - v = p.\frac{1+p^2}{q}$; wherefore, by substitution, $p.\frac{1+p^2}{q}\left\{1 - \frac{dv}{dx}\right\} - \frac{1+p^2}{q}\left\{p - \frac{dw}{dx}\right\} = \frac{\text{R}d\text{R}}{dx}$; or $-p.\frac{1+p^2}{q}\frac{dv}{dx} + \frac{1+p^2}{q}\cdot\frac{dw}{dx} = \frac{\text{R}d\text{R}}{dx}$.

Also, equation (3), $-\frac{dv}{dx} = p\frac{dw}{dx}$, or $p = -\frac{dv}{dw}$; and

$\text{R} = \frac{(1+p^2)^{\frac{3}{2}}}{-q}$; wherefore, by substitution and division, $(1+p^2)^{\frac{1}{2}}\frac{dw}{dx} = \frac{d\text{R}}{dx}$, or $d\text{R} = \sqrt{dv^2 + dw^2}$; and integrating, $\text{R} =$ the length of the evolute $\pm$ a constant quantity.

Cor. 1. Since every curve has an involute, if the radius of curvature of the involute can be found in algebraick terms, the curve may be rectified.

Cor. 2. The difference of any two radii of the involute equals the intercepted arc of the evolute.

These two propositions, combined with that of Art. 7, seem to demonstrate with as much accuracy as the nature of the subject allows, what by some writers is made matter of definition, that the involute is the same curve as that which is traced by the extremity of a string unwound from the evolute.

14. *Required the evolute of a known curve.*

Let x and y be the co-ordinates of the involute,
v and w the corresponding co-ordinates of the evolute.

Then it is shown in the former article that

$v = x - \frac{p}{q}(1+p^2)$ and $w = y + \frac{1+p^2}{q}$; which, calculated

in terms of x from the equation of the curve, will give the required equation between v and w, provided that x can be eliminated.

Ex. 1. Required the evolute of a parabola.

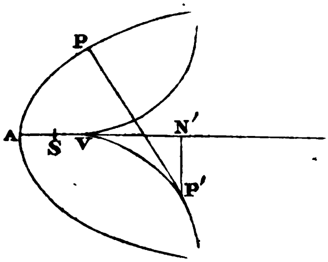
$$y = a^{\frac{1}{2}}x^{\frac{1}{2}} \therefore p = \frac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{2}} \therefore q = -\frac{1}{4}a^{\frac{1}{2}}x^{-\frac{3}{2}} \therefore$$

$$v = x + 2x\left(1 + \frac{a}{4x}\right) = 3x + \frac{a}{2} \therefore x = \frac{1}{3}\left(v - \frac{a}{2}\right).$$

$$\text{Also, } w = a^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{4x+a}{4x}\,\frac{4x^{\frac{3}{2}}}{a^{\frac{1}{2}}} = -\frac{4x^{\frac{3}{2}}}{a^{\frac{1}{2}}} \therefore x = -\sqrt[3]{\frac{aw^2}{16}};$$

$$\text{whence } \frac{1}{3}\left(v - \frac{a}{2}\right) = -\sqrt[3]{\frac{aw^2}{16}} \therefore w^2 = \frac{16}{27a}\left(v - \frac{a}{2}\right)^3$$

and $w = \dfrac{-4}{3\sqrt{3a}}\left(v - \dfrac{a}{2}\right)^{\frac{3}{2}}$; the equation of a semi-cubical parabola, the negative sign showing that it lies below the axis of its involute.



When $v = \frac{a}{2}$, $w = 0$; take therefore AV = 2AS, and v is the vertex of the evolute.

Transfer the origin to V, by substituting $v' = v - \frac{a}{2}$, and we have $w^2 = \dfrac{16}{27a}v'^3$ for the equation of the curve VP'.

Ex. 2. The evolute of an ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \therefore \frac{x}{a^2} + \frac{py}{b^2} = 0 \therefore \frac{1}{a^2} + \frac{p}{b^2} + \frac{qy}{b^2} = 0;$$

$$\text{whence } p = -\frac{b^2x}{a^2y}, \text{ . . . .}$$

$$\text{and } q \text{ which} = -\frac{b^2}{y}\left\{\frac{1}{a^2} + \frac{p^2}{b^2}\right\} = -\frac{b^2}{a^2y} - \frac{b^4x^2}{a^4y^3} \text{ . . . .}$$

$$= -\frac{a^2b^4\left\{\dfrac{y^2}{b^2} + \dfrac{x^2}{a^2}\right\}}{a^4y^3} = -\frac{b^4}{a^2y^3}.$$

Now $w = y + \frac{1+p^2}{q}$ $\therefore qw = qy + 1 + p^2$ $\therefore$ by substitution, $-\frac{b^4}{a^2y^3}w = -\frac{b^4}{a^2y^2} + 1 + \frac{b^4x^2}{a^4y^2} = -\frac{b^4}{a^2y^2} + 1 + \frac{b^4}{a^2y^2}\left\{1 - \frac{y^2}{b^2}\right\} = 1 - \frac{b^2}{a^2} = \frac{a^2-b^2}{a^2}$ $\therefore bw = -\frac{y^3}{b^3}.(a^2-b^2)$ and $(bw)^{\frac{2}{3}} = \frac{y^2}{b^2}.(a^2-b^2)^{\frac{2}{3}}$.

Also, $v = x - \frac{p}{q}(1+p^2) = x - \frac{y^2x}{b^2}\left(1 + \frac{b^4x^2}{a^4y^2}\right) \ldots\ldots$ $= x - \frac{b^2x^3}{a^4} - \frac{y^2x}{b^2} = x - \frac{b^2x^3}{a^4} - x\left\{1 - \frac{x^2}{a^2}\right\} = \frac{x^3}{a^2} - \frac{b^2x^3}{a^4}$

$\therefore av = \frac{x^3}{a^3}.(a^2-b^2)$ and $(av)^{\frac{2}{3}} = \frac{x^2}{a^2}.(a^2-b^2)^{\frac{2}{3}}$.

Wherefore $(av)^{\frac{2}{3}} + (bw)^{\frac{2}{3}} = \frac{x^2}{a^2}.(a^2-b^2)^{\frac{2}{3}} + \ldots\ldots$ $\frac{y^2}{b^2}.(a^2-b^2)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}$, the equation of the evolute.

Cor. The equation of the evolute of an hyperbola is $(av)^{\frac{2}{3}} - (bw)^{\frac{2}{3}} = (a^2+b^2)^{\frac{2}{3}}$, which, in the case of the equilateral hyperbola, becomes $v^{\frac{2}{3}} - w^{\frac{2}{3}} = (2a)^{\frac{2}{3}}$.

Ex. 3. A cycloid.

Place the origin at the centre of the base; then (9.8. Ex. 1.)

$$p = -\sqrt{\frac{x}{2a-x}} = -(2ax^{-1}-1)^{-\frac{1}{2}} \therefore q = \frac{-ax^{-2}}{(2ax^{-1}-1)^{\frac{3}{2}}}$$

$$= \frac{-a}{x^{\frac{1}{2}}(2a-x)^{\frac{3}{2}}} \therefore \frac{p}{q} = \frac{x(2a-x)}{a}; \text{ and } 1+p^2 = \frac{2a}{2a-x} \therefore,$$

by substitution, $v = x - 2x = -x$.

$$\text{Also, } w = y - \frac{2a}{2a-x} \div \frac{a}{x^{\frac{1}{2}}(2a-x)^{\frac{3}{2}}} = y - 2\sqrt{2ax-x^2};$$

but $y = a\,vs.^{-1}\frac{2a-x}{a} + \sqrt{2ax-x^2}$ $\therefore w = a\,vs.^{-1}\frac{2a-x}{a} - \sqrt{2ax-x^2}$; in which, if $-v$ be substituted for x, there results $w = a\,vs.^{-1}\frac{2a+v}{a} - \sqrt{-2av-v^2}$, which shows that

v is necessarily negative; or the required equation is $w = a\, vs.^{-1} \frac{2a-v}{a} - \sqrt{2av - v^2}$, where v is to be measured in an opposite direction from x.

If this equation be transferred along the axis Y through $2\pi a$, there results $w = a(2\pi - vs.^{-1} \frac{2a-v}{a}) + \sqrt{2av - v^2} =$

$a.\, vs.^{-1} \frac{v}{a} + \sqrt{2av - v^2}$, which is the same curve as the involute, the origin being at the vertex.

The cycloid is a rectifiable curve, since its radius of curvature can be obtained in algebraical terms.

15. *Required the involute of a known curve.*

Make the same substitutions as in the former article. Differentiate $R^2 = (x - v)^2 + (y - w)^2$ on the supposition that v and w vary, in which case all the magnitudes necessarily vary; wherefore $RdR = -(x - v)dv - (y - w)dw + x - v + (y - w) = -(x - v)dv - (y - w)dw$, since (1) $x - v + (y - w)p = 0$.

Also, it may be shown as in Art. 13, that $p = -\frac{dv}{dw}$*;

whence by substitution in (1) $x - v - \frac{dv}{dw}(y - w) = 0$, or

$(x - v)dw - (y - w)dv = 0$; which combined with $RdR = -(x - v)dv - (y - w)dw$ will give x and y in terms of v, w, dv, dw and R, dR.

To eliminate R, we have $R = \sqrt{(x - v)^2 + (y - w)^2}$, and (Art. 12.) $dR = \sqrt{dv^2 + dw^2}$; whence, by substitution, there will result a fluxional equation between v and w.

In this process we obtain an expression for dR but not for R in terms of v and w; and the reason is, that the same evolute may have an infinite number of involutes depending upon the length of the string which is unwound.

16. *Required the evolute of a known spiral.*

Let r and p = the radius vector and the perpendicular on the tangent in the involute.

* This also follows from the circumstance that the fluxional triangles of the involute and evolute are similar.

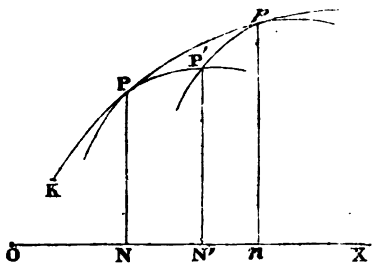
ρ and π = the same lines in the evolute.

R = the radius of curvature of the involute, which may be found in terms of r and p.

(Eu. 47. 1.) $\rho^2 = r^2 - p^2 + (\text{R} - p)^2$, and $\pi^2 = r^2 - p^2$; from which R, r, and p may be eliminated, and the equation found between ρ and π.

If the evolute be given, R is now the length of the curve; and a fluxional equation may be obtained for the involute.

17. *Required to determine the curve which touches any number of curves of a given species, described after a certain law.*



Let PP', P'p be two positions of the moveable curve near to each other, and intersecting in P'; KPp the required curve which touches them both.

Let $\text{F}(x, y, a) = 0 = u$ be the equation of PP'; a being the parameter by the variation of which it comes into the position P'p; and x, y being the co-ordinates of any point in PP'. When PP' becomes P'p, let a become A; then, since the species of the curve is given, the equation of P'p is $\text{F}(x, y, \text{A}) = 0$; x, y being now the co-ordinates of any point in P'p.

Now diminish $\text{A} - a$ indefinitely; then p, and à fortiori P', which is always between P and p, if Pp touches both of them, approximates to P as its limit. If then we differentiate the original equation $\text{F}(x, y, a) = 0$ on the supposition that a varies and that x and y are constant, and eliminate a by combining this equation $\frac{du}{da} = 0$ with $u = 0$, the values of x and y in the resulting equation belong to a point which is the limit of the intersections of the two curves, and therefore belong to P; or it is the required equation.

If the equation $u = 0$ has two variable parameters a and b, whose relation is given by the law after which the curves are described, in the differentiation b must be differentiated as a function of a.

18. *Otherwise.* The moveable and the tangential curve

must have the same co-ordinates, because the points of contact are common to both; and the only difference between them is that the one is generated on the supposition that the parameter is constant; and the other, that it varies. Hence, if $u = \text{F}(x, y, a) = 0$ is the equation of the moveable curve, differentiating on the two suppositions, we shall have $du = \frac{du}{dx}dx + \frac{du}{dy}dy = 0$, and $du = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{da}da = 0$. Now these equations are *simultaneous*, because there must result from each the same value of $\frac{dy}{dx}$, which marks the position of the tangent; wherefore $\frac{du}{da} = 0$; and this combined with $u = 0$ will give the required equation of the tangential curve.

19. *Examples.*

Ex. 1. The centre of a given circle moves along a given right line; required the line which touches all the circles.

Let a and b be the co-ordinates of the centre of one of the circles; r the radius.

$b = ma + \mu$ the given right line; then $r^2 = (x - a)^2 + (y - b)^2$ is the equation of the moveable curve $\therefore \frac{du}{da} = 0 = (x - a) + (y - b)\frac{db}{da} = x - a + m(y - b)$ (1) $\therefore$, by substitution, $r^2 = \frac{m^2+1}{m^2}(x - a)^2 \therefore x - a = \frac{mr}{\sqrt{m^2+1}} \ldots$ $\therefore a = x - \frac{mr}{\sqrt{m^2+1}}$ and b, which $= ma + \mu$, $= mx - \frac{m^2r}{\sqrt{m^2+1}} + \mu$ $\therefore$, by substitution in (1), the required equation is $0 = \frac{mr}{\sqrt{m^2+1}} + my - m^2x + \frac{m^3r}{\sqrt{m^2+1}} - m\mu$, or $y = mx + \mu - \sqrt{m^2 + 1} \,.\, r$, which is a line parallel to the given line.

Ex. 2. Required the nature of the curve in which the normal varies as its distance from a given point in the axis.

The normal may be supposed to be the radius of a variable circle whose centre moves along the axis; in which case the required curve touches all the circles.

Let a = the abscissa of the foot of the normal,
b = the abscissa of the given point,
e = the constant ratio.

$$(a-x)^2 + y^2 = r^2 = e^2(a-b)^2 \therefore \frac{du}{da} = a - x - e^2(a-b) = 0$$

$$\therefore a = \frac{x - e^2 b}{1 - e^2} \therefore a - x = \frac{e^2(x-b)}{1-e^2}, \text{ and } a - b = \frac{x - b}{1 - e^2}$$

$\therefore$ by substitution and transposition, $y^2 = e^2 \dfrac{(x-b)^2}{(1-e^2)^2} - \dfrac{e^4(x-b)^2}{(1-e^2)^2} = \dfrac{e^2(x-b)^2}{1-e^2} \therefore y = \dfrac{e(x-b)}{\sqrt{1-e^2}}$, or the required curve is a right line.

Ex. 3. Required the curve which a right line of given length, intercepted by two rectangular co-ordinates, perpetually touches.

The equation of the line is $\dfrac{x}{a} + \dfrac{y}{b} = 1$; let c = its length, then $a^2 + b^2 = c^2$.

Differentiating, $\dfrac{x}{a^2} + \dfrac{y}{b^2}\dfrac{db}{da} = 0$; but $\dfrac{db}{da} = -\dfrac{a}{b}$ $\therefore$

$\dfrac{x}{a^3} - \dfrac{y}{b^3} = 0 \therefore \dfrac{1}{b} = \dfrac{1}{a} \cdot \dfrac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}}$ $\therefore$ by substitution, $\dfrac{x}{a} + \dfrac{x^{\frac{1}{3}}y^{\frac{2}{3}}}{a} = 1 \therefore a = x + x^{\frac{1}{3}}y^{\frac{2}{3}}$, and $b = \dfrac{ay^{\frac{1}{3}}}{x^{\frac{1}{3}}} = x^{\frac{2}{3}}y^{\frac{1}{3}} + y$ $\therefore$

$$c^2 = a^2 + b^2 = (x + x^{\frac{1}{3}}y^{\frac{2}{3}})^2 + (y + x^{\frac{2}{3}}y^{\frac{1}{3}})^2 = (x^{\frac{2}{3}} + y^{\frac{2}{3}})^3$$

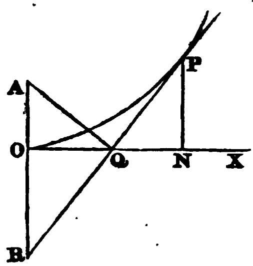
or $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$ is the required equation.

Ex. 4. One of the angles of a triangle is constant, and also the product of the sides which contain it; required the curve to which the third side of the triangle is always a tangent.

The equation of the third side is $\dfrac{x}{a} + \dfrac{y}{b} = 1$, where the co-ordinates are not necessarily rectangular.

Let $ab = c^2$; then, differentiating, $\frac{x}{a^2} + \frac{y}{b^2}\frac{db}{da} = 0$; but $\frac{db}{da} = -\frac{c^2}{a^2} = -\frac{b}{a}$ $\therefore$ by substitution and reduction, . . . $\frac{x}{a} - \frac{y}{b} = 0$ $\therefore \frac{2x}{a} = 1$ $\therefore a = 2x$ and $b = 2y$ $\therefore 4xy = ab = c^2$, which is the equation of an hyperbola.

Ex. 5. A right line and a point are given in position; lines are drawn intersecting the given line under the condition that their normals drawn from the points of intersection shall pass through the given point; required the curve to which these lines are always a tangent.



From the given point A draw AO perpendicular to the given line ONX; place the origin at O; let PQR be a line intersecting OX in Q, join AQ which is at right angles to PQ.

Let AO $= c$ and let $\frac{x}{a} - \frac{y}{b} = 1$ be the equation of PQR; then, since AQR is a right angle, $a^2 = bc$.

Differentiating, $\frac{x}{a^2} - \frac{y}{b^2}\frac{db}{da} = 0$; but $\frac{db}{da} = \frac{2a}{c} = \frac{2b}{a}$

$\therefore$ by substitution and reduction, $\frac{x}{a} - \frac{2y}{b} = 0$ $\therefore \frac{y}{b} = 1$, or $b = y$ and $\frac{x}{a} = 2$, or $a = \frac{x}{2}$ $\therefore \frac{x^2}{4} = yc$, or $x^2 = 4cy$; which shows that the curve is a parabola, vertex O and focus A. (Leybourn's Math. Rep. vol. 3. p. 73.)

Ex. 6. A fluid issues from a vertical cylinder with innumerable holes; required the surface which bounds the issuing fluid.

Let $a =$ the depth of one of the holes, then $y^2 = 4a(x-a)$ is the curve described by the issuing particle; $\therefore \frac{du}{da} = 0 = x - 2a$, or $a = \frac{x}{2}$ $\therefore$ the required

equation is $y^2 = 2x.\dfrac{x}{2} = x^2$, or $y = \pm x$; which shows that the required surface is the frustum of a cone whose vertical angle $= 45°$.

Ex. 7. Required the surface which bounds all the parabolas described from a given point with a given velocity of projection.

Let h = the space due to the velocity of projection,
θ = the $\angle$ of projection of one of the parabolas.

Then the equation is $y = x \tan.\theta - \dfrac{x^2}{4h}\sec.^2\theta$, and considering θ as the variable parameter, $\dfrac{du}{d\theta} = 0 = x\sec.^2\theta - \dfrac{x^2}{2h}\sec.^2\theta\tan.\theta \quad \therefore \tan.\theta = \dfrac{2h}{x}$, and $\sec.^2\theta = 1 + \dfrac{4h^2}{x^2} \quad \therefore$ by substitution, the required equation is . . . .
$y = 2h - \dfrac{x^2}{4h}\left(1 + \dfrac{4h^2}{x^2}\right) = h - \dfrac{x^2}{4h} \quad \therefore x^2 = 4h(h-y)$; or the required surface is that of a paraboloid whose focus is the point of projection, and whose latus rectum $= 4h$.

Ex. 8. Required the nature of the curve in which the axis intercepted between the tangent and the vertex varies as the nth power of the perpendicular to the axis drawn from the vertex and terminated by the tangent.

The equation of the tangent is $-\dfrac{x}{a} + \dfrac{y}{b} = 1$. Assume $a = b^n$, omitting for the present the constant c which renders the equation homogeneous.

$$\frac{db}{da} = \frac{1}{nb^{n-1}} \quad \therefore \text{differentiating}, \quad -\frac{x}{a^2} + \frac{y}{b^2}\cdot\frac{1}{nb^{n-1}} = 0$$

$$\therefore \frac{a^2}{x} = \frac{nb^{n+1}}{y} \quad \therefore a = \left(\frac{nx}{y}\right)^{\frac{1}{2}} b^{\frac{n+1}{2}} = b^n \quad \therefore b = \left(\frac{nx}{y}\right)^{\frac{1}{n-1}}$$

and $a = \left(\dfrac{nx}{y}\right)^{\frac{n}{n-1}} \quad \therefore$ by substitution, $-x\left(\dfrac{y}{nx}\right)^{\frac{n}{n-1}}$ . . . .

$$+ y\left(\frac{y}{nx}\right)^{\frac{1}{n-1}} = 1, \text{ or } -\frac{y^{\frac{n}{n-1}}}{n^{\frac{n}{n-1}}x^{\frac{1}{n-1}}} + \frac{y^{\frac{n}{n-1}}}{n^{\frac{1}{n-1}}x^{\frac{1}{n-1}}} = 1, \text{ or}$$

$$\frac{x^{\frac{1}{n-1}}}{y^{\frac{n}{n-1}}} = \frac{1}{n^{\frac{1}{n-1}}} - \frac{1}{n^{\frac{n}{n-1}}} = \frac{n-1}{n^{\frac{n}{n-1}}} \therefore x = \frac{(n-1)^{n-1}}{n^n} y^n \text{; or,}$$

introducing c, $c^{n-1}x = \dfrac{(n-1)^{n-1}}{n^n} y^n$ is the required equation.

20. *Required the nature of the curve which cuts at a given angle any number of curves of a given species described after a certain law.*

Def. The required curve in this and the former article, in order to distinguish it from the moveable curve, is called the *trajectory*.

Let θ and α = the angles which the tangent of the moveable curve makes with the axis and the trajectory respectively,
θ' = the angle which the tangent of the trajectory makes with the axis,
x and y = the co-ordinates of the moveable curve.
x' and y' = those of the trajectory.

$$\text{Then } \theta - \theta' = \alpha \therefore \tan.\alpha = \frac{\tan.\theta - \tan.\theta'}{1 + \tan.\theta \tan.\theta'} = \frac{\frac{dy}{dx} - \frac{dy'}{dx'}}{1 + \frac{dy}{dx}\frac{dy'}{dx'}}.$$

Also, let $\text{F}(x, y, a) = 0$ be the given equation, a being the parameter upon the value of which the position of the moveable curve depends. By differentiation, the value of $\frac{dy}{dx}$ may be obtained from this equation in terms of x, y and a; and substituting this value in the expression for $\tan.\alpha$, a may be eliminated, and there will result a fluxional equation which belongs to the trajectory; for, since the curves intersect, x and y are the same as x' and y'.

Cor. If the given $\angle = 90°$, or the trajectory is rectangular, a is eliminated by means of the equation . . . . . .
$$1 + \frac{dy}{dx}\frac{dy'}{dx'} = 0.$$

Problems of this class necessarily involve the integration of a fluxional equation; and as we must return to the subject in the second volume we shall subjoin but few examples, and a short praxis of this and the preceding article.

Ex. 1. Let $y = ax$; required the nature of the curve which intersects all the lines to which this equation can belong at right angles.

Here $\frac{dy}{dx} = a = \frac{y}{x}$ $\therefore$ by substitution, the fluxional equation of the trajectory is $1 + \frac{y}{x}\frac{dy}{dx} = 0$, or $xdx + ydy = 0$ $\therefore$ integrating $x^2 + y^2 = c^2$, or the trajectory is a circle.

Ex. 2. The moveable curve is a semicubical parabola; required the nature of the curve which, beginning at the vertex of the parabola in its first position, cuts it at right angles. (Cambridge Problems, p. 293).

$ax^2 = y^3$ $\therefore \frac{dy}{dx} = \frac{2ax}{3y^2} = \frac{2y}{3x}$ $\therefore$ by substitution, $1 + \frac{2ydy}{3xdx} = 0$ $\therefore \frac{3x^2}{2} + y^2 = c^2$, or $\frac{x^2}{2} + \frac{y^2}{3} = c^2$ which is an ellipse whose axes are in the ratio of $\sqrt{2} : \sqrt{3}$.

21. PRAXIS.

1. Required the nature of the curve in which (normal)$^2 \propto$ its distance from the vertex. If $r^2 = b \times a$, the required curve is a parabola, whose latus rectum $= b$; and the origin the focus.

2. One of the angles of a triangle is fixed, and the sum of the sides which contain it is constant; required the curve to which the third side is always a tangent. The required equation is $x^{\frac{1}{2}} \pm y^{\frac{1}{2}} = c^{\frac{1}{2}}$ according as the sum or the difference of the sides is given: and the required curve is consequently a parabola.

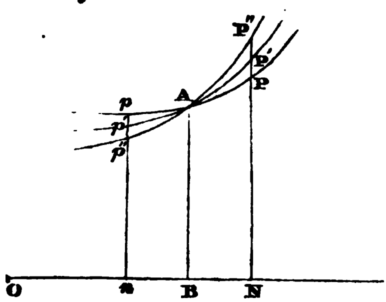
3. One of the angles of a triangle is fixed, and the area included by the third side is constant; required the curve to which the third side is always a tangent. The required curve is an hyperbola between the asymptotes.

4. Two points move from given positions with equal uniform velocities at right angles to each other; required the curve to which the line which joins them is always a tangent $x^{\frac{1}{2}} + y^{\frac{1}{2}} = (m + n)^{\frac{1}{2}}$.

5. Investigate the nature of the curve in which lines drawn from a given point perpendicular to the tangent may always be equal. A circle.

6. Required the curve, the centre of whose curvature describes the axis.

22. *On the different orders of contact.*



Def. Let there be two curves PAp, $P'Ap'$, which intersect in A; draw A's co-ordinates OB, BA, and the contiguous and equidistant ordinates NPP', $np'p$.

Let $OB = x$, $BA = y$, $BN = Bn = h$; then, developing the ordinates by Taylor's theorem, let

$$NP = y + A \cdot \frac{h}{1} + B \cdot \frac{h^2}{1.2} + C \cdot \frac{h^3}{1.2.3} + \&c.$$

$$NP' = y + A_1 \cdot \frac{h}{1} + B_1 \cdot \frac{h^2}{1.2} + C_1 \cdot \frac{h^3}{1.2.3} + \&c.$$

First suppose $A = A_1$, then (8. 2) AP and AP' have a common tangent at A; and this is called simple contact, or *contact of the first order.* Next suppose that B also $= B_1$, then the curves are said to have a *contact of the second order;* if C also $= C_1$, the contact is of the *third order;* and generally, the contact is of the nth order, if the first fluxional coefficients, which are not equal, are of the $(n + 1)$th order.

Cor. Of the equation of a right line all the fluxional coefficients vanish after the first; and consequently a right line cannot have a more intimate contact than the first with any proposed curve, except at certain points where, by assigning particular values to the principal variable, the second fluxional coefficient of the proposed curve vanishes. In this case the contact is of the second order.

23. *If the order of the contact is even, there is both contact and intersection; if odd, only contact.*

For supposing the curves to intersect as in the former article, h may be assumed so small that the magnitude of the ordinates shall be in the order of A, A_1; which therefore determine the position of AP, AP' with respect to each other and to the axis. Also by the same theorem . . . . .

$$\left.\begin{aligned} np &= y - \text{A}\,.\frac{h}{1} + \text{B}\,.\frac{h^2}{1.2} - \text{C}\,.\frac{h^3}{1.2.3} + \&\text{c.} \\ np' &= y - \text{A}_1\,.\frac{h}{1} + \text{B}_1.\frac{h^2}{1.2} - \text{C}_1.\frac{h^3}{1.2.3} + \&\text{c.} \end{aligned}\right\}$$ which are therefore in the reverse order of A, A_1 as they ought to be, since the curves cross at A.

First, suppose $\text{A} = \text{A}_1$, but B not $= \text{B}_1$; or that the contact is of the *first* order; then the order of the magnitudes np, np' is the same as that of NP, NP', and consequently there is contact but not intersection. Next let $\text{A} = \text{A}_1$, $\text{B} = \text{B}_1$, and C not $= \text{C}_1$; here the order of the magnitudes np, np' is the reverse of that of NP, NP'; and consequently contact of the *second* order is both contact and intersection. In the same manner it may be shown that all contacts of an even order are, and that those of an odd order are not intersections.

24. Let AP, AP' have contact of any order, the third for instance; draw any other curve as AP'' through A, the developement of whose ordinate is

$$\text{NP}'' = y + \text{A}_2\,.\frac{h}{1} + \text{B}_2.\frac{h^2}{1.2} + \text{C}_2\,.\frac{h^3}{1.2.3} + \&\text{c.};$$ then it is manifest that AP'' cannot pass between AP and AP', unless $\text{A}_2 = \text{A}_1$, $\text{B}_2 = \text{B}_1$, and $\text{C}_2 = \text{C}_1$. If these equations obtain, the position of AP'' with respect to AP and AP' depends upon the order of the magnitudes D, D_1, D_2.

Generally, it is obvious that "a curve drawn through the point of contact of two other curves cannot pass between them, unless it touches one of them at least as intimately as they touch each other; in which case the relative position of the three curves will depend upon the magnitude of the first coefficients which are not equal."

Cor. If all the coefficients are equal the curves coincide.

25. *Required to investigate the different orders of contact which two curves, whose species are given, can bear to each other.*

Let U and V represent the equations of the two curves, of which U contains m; and V, n arbitrary constants; suppose m to be not less than n, and that none of the fluxional coefficients vanish.

Differentiate both equations $n-1$ times; equate the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \dots : \frac{d^{n-1}y}{dx^{n-1}}$; which will give n equations, by means of which the constants in V may be expressed in terms of those of U; which determines the curve of the

species v which has contact of the $(n-1)$th order with any proposed curve of the species U.

It is manifest that there cannot be an higher contact than the $(n-1)$th except the nth coefficients of U and V are equal. As this may happen at particular points, the enunciation of the proposition must be limited as follows. *A curve of a given species,* in general, *cannot have with any proposed curve contact of an higher order than the number of the constants contained in its equation diminished by unity.*

If we equate only $n-1$ terms $y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \ldots \frac{d^{n-2}y}{dx^{n-2}}$, there will remain a constant in V which is undetermined; which shows that there may be an infinite number of curves of the species V, which has contact of the $n-2$th order with the proposed curve.

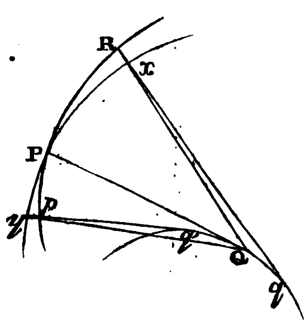
Cor. 1. The contact of a circle with any proposed curve, in general, cannot be of an higher order than the second; but if there are points in the proposed curve at which the third fluxional coefficients of the ordinates are equal, the contact at these points is of the third order.

Cor. 2. It appears from Art. 2, that that circle which has contact of the second order with any proposed curve is the same as its circle of curvature.

Cor. 3. The circle of curvature and the curve which it touches have, in general, both contact and intersection. At those points where the third fluxional coefficients are equal, there is contact without intersection, provided that the fourth fluxional coefficients are not also equal. (Art. 23.)

26. *Required to demonstrate, geometrically, that a curve and its circle of curvature have, in general, both contact and intersection.*

Let RP$'p$ be the curve, qQq' its evolute, xPy the circle of curvature at P, whose radius is QP; then taking Qq a small arc of the evolute and joining Qx, we have (Eu. 1. 20.) xq less than xQ + Qq, or PQ + Qq, or Rp; or R falls without the circle. Similarly it may be shown that p falls within the circle, or the curve intersects the circle in P.



27. *At the points of the greatest and least curvature, the contact of the circle of curvature is of the* third *order.*

For, let p, q, r be the fluxional coefficients of the curve; p, q, r' those of the circle of curvature.

Let R = the radius of curvature, then $R = \frac{(1+p^2)^{\frac{3}{2}}}{-q}$, and when the curvature is a maximum or minimum, ...

$0 = -3(1+p^2)^{\frac{1}{2}}p + \frac{(1+p^2)^{\frac{3}{2}}r}{q^2}$, or . . . .

$(1+p^2)^{\frac{1}{2}}\left\{(1+p^2)r - 3pq^2\right\} = 0$, and $r = \frac{3pq^2}{1+p^2}$.

Also, from the equation of the circle $(x-\alpha)^2+(y-\beta)^2 = R^2$, therefore differentiating, $x-\alpha+(y-\beta)p = 0$, and $1+p^2+(y-\beta)q = 0$, and $2pq+pq+(y-\beta)r' = 0$, or $r' = \frac{3pq}{y-\beta} = \frac{3pq^2}{1+p^2} = r$; or the contact is of the third order.

Cor. At the points of the greatest and least curvature, the evolute has what is called a *cusp* (Ch. 12. 11.) and the demonstration of the preceding article fails at these points.

Ex. At the extremity of the axis minor of an ellipse, $p = 0$, $q = -\frac{b}{a^2}$, $r = 0$, both in the curve and in the circle of curvature; but $s = -\frac{3b}{a^4}$, and $s' = -\frac{3b^3}{a^6}$; and since we may suppose b to be greater than a, it follows that the contact is of the third order, at the extremities of both the axes.

Since $s'-s = \frac{3b}{a^6}(a^2-b^2)$, the circle of curvature at the extremity of the axis minor falls wholly without; and at the extremity of the axis major, it falls wholly within the curve.

28. *Def.* That curve which touches any proposed curve so intimately that no other of the same species can be drawn between them through the point of contact is called its *osculating curve* of that species.

Ex. 1. Since a right line admits in general of only simple contact, no other right line can be drawn between a curve and its tangent which does not cut the curve (25.); or the tangent is a line which has the first order of osculation. If the

second fluxional coefficient of the curve vanishes the tangent has the second order of oscillation. In this case there is both contact and intersection.

Ex. 2. Since no touching circle can be drawn between a curve and its circle of curvature, the circle of curvature is the osculating circle; it has osculation of the second order, except at particular points where the osculation may be of the third order.

Ex. 3. The different degrees of osculation are most conveniently measured by a series of curves whose equations are of the form $y = a + bx + cx^2$, $y = a + bx + cx^2 + ex^3$, ... $y = a + bx + cx^2 \ldots . qx^n$. These equations belong to parabolick and not hyperbolick curves, as they do not admit of a rectilinear asymptote.

Now it is manifest that there is an infinite number of curves of the first species which have contact of the first order with any proposed curve; but that there is only one which, in general, has contact of the second order. This then is called the *osculating parabola of the second order*: it is an Apollonian parabola; and the constants are to be determined by elimination from the equation of the proposed curve. In the same manner, there is an infinite number of curves of the second species which have contact of the first or of the second order with the proposed curve, but there is only one which has contact of the third order; and when the constants are determined as before by elimination, it is the *osculating parabola of the third order*: and thus there may be found a series of parabolick curves which have all possible degrees of osculation with the proposed curve.

The process of elimination is facilitated by transferring the origin of the parabolick curve to the point of contact.

Osculating parabolas of an even order both touch and intersect; those of an odd order only touch the curve (Art. 23).

It may be observed that $y = a + bx + cx^2$ is not the canonical equation of the Apollonian parabola, which contains four constants; one which determines its magnitude; two, the position of the vertex; and the fourth, the position of the axis. In this equation the position of the axis is fixed; it is parallel to Y.

To exemplify the method of finding these curves; let it be required to draw to any point of a proposed curve the osculating parabola of the second order.

The required equation, by transferring the origin to the point of contact, takes the form of $y = bx + cx^2$, or $y = c\left(x^2 + \frac{bx}{c}\right)$, or $y + \frac{b^2}{4c} = c\left(x + \frac{b}{2c}\right)^2$; and consequently the co-ordinates of the vertex are $-\frac{b^2}{4c}$ and $-\frac{b}{2c}$, and the latus rectum $= \frac{1}{c}$.

Differentiating, $p = b + 2cx$, $q = 2c$; wherefore at the point of contact $b = p$, $c = \frac{q}{2}$; or the required curve is an Apollonian parabola, whose axis is parallel to Y; the co-ordinates of whose vertex on the axes Y and X, the point of contact being the origin, are $-\frac{p^2}{2q}$ and $-\frac{p}{q}$; and whose latus rectum $= \frac{1}{2q}$; where the values of p and q belonging to the point of contact are to be determined from the equation of the given curve.

29. *Required to represent geometrically the fluxional coefficients of* y = fx, *the equation of any proposed curve.*

By Taylor's theorem $y' = y + p \cdot \frac{h}{1} + q \cdot \frac{h^2}{1.2} + r \cdot \frac{h^3}{1.2.3}$ + &c.

Describe through A_1 (fig. 22) the point whose co-ordinates are x, y the successive osculating parabolas beginning with the tangent which may be considered as the osculating parabola of the first order; draw the ordinate NPP′P″ .... near to AB, cutting the curve in P, and the tangent and the succeeding parabolas in P′, P″, P‴, ...; then we have

$$\left.\begin{aligned} NP &= y \\ NP' &= y + p \cdot \frac{h}{1} \\ NP'' &= y + p \cdot \frac{h}{1} + q \cdot \frac{h^2}{1.2} \\ NP''' &= y + p \cdot \frac{h}{1} + q \cdot \frac{h^2}{1.2} + r \cdot \frac{h^3}{1.2.3} \\ \&c. &= \&c. \end{aligned}\right\} \text{whence, there results}$$

$$\left.\begin{aligned} PP' &= p \cdot \frac{h}{1} \\ P'P'' &= q \cdot \frac{h^2}{1.2} \\ P''P''' &= r \cdot \frac{h^3}{1.2.3} \\ \&c. &= \&c. \end{aligned}\right\}$$

which lines, therefore, represent the linear magnitude of the terms of Taylor's theorem; and if h be assumed $= dx$, we shall have $PP' = dy$, $2P'P'' = d^2y$, $2.3P''P''' = d^3y$ ... geometrical representations of the successive fluxions of the ordinate.

30. *If there is a point of a curve, at which an Apollonian parabola may have contact of the fourth order; at the two points contiguous to it, the osculating curves of the fourth order are respectively an ellipse and an hyperbola.*

The canonical equation of the Apollonian parabola contains four arbitrary constants, and consequently that Apollonian parabola may be determined which has, in general, the third; and which at particular points may have the fourth order of contact.

Let $ay^2 + (b + cx)y + e + fx + gx^2 = 0$ be the canonical equation of the second degree; then, since it contains five arbitrary constants (7. 19. Ex. 3.), that conick section may be determined which has contact of the fourth order with the proposed curve; but the conick section to which the canonical equation belongs depends upon the sign of $4ag - c^2$ (Alg. 508), it being an ellipse or an hyperbola according as the sign is + or −; and a parabola if $4ag - c^2 = 0$. Hence, calculate, as in article 28, the value of $4ag - c^2$ in terms of x and y, and the real roots of the resulting equation $4ag - c^2 = 0$ will give those points of the curve at which an Apollonian parabola has contact of the fourth order. At the two points contiguous to these particular points, since $4ag - c^2$ must change its sign in passing through zero, the osculating curves are in the one case an ellipse, and in the other an hyperbola.

CHAPTER XII.

Singular points of Curves.

1. *Def.* AN *inflexion*, or point of *contrary flexure*, is that point of a curve at which it ceases to be concave and begins to be convex to the axis.

This definition does not apply at the point where the curve intersects the axis; a circumstance which can always be ascertained from the curve's equation.

2. *The second fluxion of the ordinate of a* convex *curve is* positive, and of a concave *curve* negative; *the abscissa of the curve being supposed to increase.*

The characteristick property of a convex curve is, that its ordinate increases faster; and of a concave curve, that it increases slower than the corresponding ordinate of the tangent.

Let $y = fx$ be the curve's equation; and when x becomes $x' = x + h$, let y become y', and the ordinate of the tangent y''.

By Taylor's theorem $y' - y = p.\frac{h}{1} + q.\frac{h^2}{1.2}$; for h may be assumed so small that the sum of the remaining terms shall not affect the sign of the result, (3. 3.)

Also (8. 3.), the equation of the tangent is $y'' - y = p(x' - x) = p.h$; wherefore the ordinate of the curve increases faster or slower than that of the tangent, *i. e.* the curve is convex or concave according as q is positive or negative.

If the ordinate decreases, the demonstration is the same, except that p is negative in both the equations.

3. If the abscissa and ordinate both become negative, in consequence of the curve intersecting the axis, and the origin be placed at the point of intersection; and we suppose the *negative* abscissa to increase; since this will not affect the sign of h^2, the same proposition obtains; hence it follows, generally, that "the curve is convex when the ordinate and

its second fluxion have the same sign, and concave when their signs are different."

4. *Required to demonstrate Art. 2, without the aid of Taylor's theorem.*

Construct the figure as in Ch. 8. Art. 2.; in πQ take $\pi\pi' = P\pi$, and draw $\pi' s' r' p'$ parallel to $\pi s p$ cutting Pr in r'.

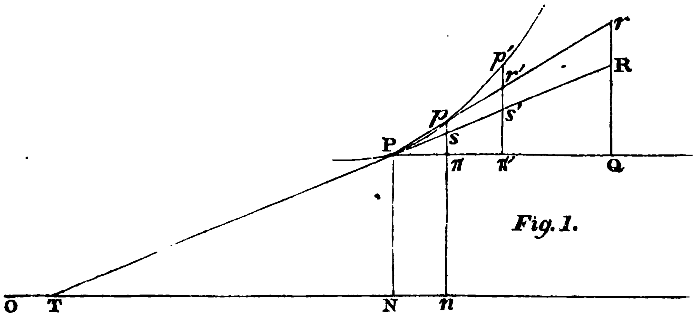


Fig. 1.

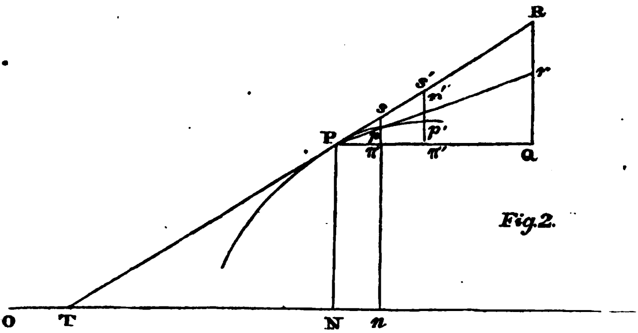


Fig. 2.

Suppose the abscissa to increase uniformly; then if the ordinate also increased uniformly, the locus of p would be a right line, and when $P\pi$ becomes $P\pi'$, $p\pi$ would become $r'\pi'$. But when the curve is convex $p'\pi'$ is greater, and when concave less than $r'\pi'$; or the ordinate increases with an accelerated velocity in the one case and with a diminished velocity in the other; and consequently in the former case its second fluxion is positive, and in the latter negative.

The curve being convex to the axis in the case of a minimum ordinate, and concave in the case of a maximum; we

see the reason why q is positive at a minimum and negative at a maximum value. (Vid. Ch. 6. Art. 4.)

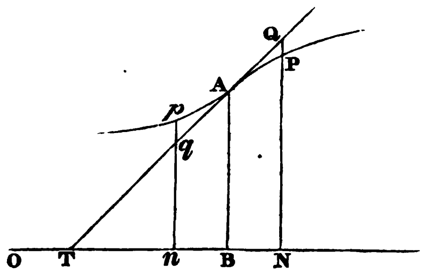
5. *The abscissa increasing, the angle which the tangent of a* concave *curve makes with the axis increases, and the distance between the origin and the tangent's intersection with the axis increases. The contrary obtains when the curve is convex.*

For $p = $ tan. $\angle$ PTN, and $\frac{dp}{dx}$ or q is positive or negative, according as the curve is convex or concave; whence tan. $\angle$ PTN, and consequently the angle itself is an increasing magnitude in the former case, and a decreasing magnitude in the latter.

Also OT $= x - \frac{y}{p}$; therefore $\frac{d.\text{OT}}{dx} = \frac{yq}{p^2}$, which is positive when the curve is convex, and negative when concave; or AT increases in the former case, and diminishes in the latter.

Cor. At an inflexion OT is a maximum or minimum.

6. *Required to investigate the conditions of a point of contrary flexure.*



The contiguous ordinates of a convex curve are both greater, and of a convex curve both less, than the corresponding ordinates of the tangent. But at an inflexion, the ordinates of the curve are the one greater, and the other less than those of the tangent. This is its characteristick property; so that there is an inflexion or not, according as this property obtains.

Let AB be any ordinate; draw the contiguous ordinates NP, np cutting the tangent at A in Q and q.

Then by Taylor's theorem and Ch. 3, Art. 3, we have

$$\left.\begin{array}{l}\text{NP} = y + p\cdot\frac{h}{1} + q\cdot\frac{h^2}{1.2}\\ np = y - p\cdot\frac{h}{1} + q\cdot\frac{h^2}{1.2}\end{array}\right\}\text{ or }\ldots\ldots\ldots\ldots\ldots$$

$$=\left\{\begin{array}{l}y + p\cdot\frac{h}{1} + q\cdot\frac{h^2}{1.2} + r\cdot\frac{h^3}{1.2.3}\\ y - p\cdot\frac{h}{1} + q\cdot\frac{h^2}{1.2} - r\cdot\frac{h^3}{1.2.3}\end{array}\right\}\text{ or } = \&\text{c.}$$

Also (8. 2) $\left.\begin{array}{l}\text{NQ} = y + p\,.\,h\\ nq = y - p\,.\,h\end{array}\right\}.$

From which it appears that unless $q = 0$, the ordinates of the curve will be either both greater or both less than those of the tangent, according as the sign of q is positive or negative, and consequently the point is not an inflexion.

Next suppose, q being $= 0$, that r is not $= 0$; here the contiguous ordinates of the curve are necessarily the one greater, and the other less than those of the tangent; and the point is an inflexion.

Again, suppose $q = 0$, $r = 0$, but s not $= 0$; in this case there will not be an inflexion; but if $q = 0$, $r = 0$, $s = 0$, but t is not $= 0$, then the point is an inflexion, and so on: whence the Rule,

"*There is an inflexion if, the fluxional coefficients* q, r, s, ... *vanishing, the first which does not vanish is of an* odd *number of dimensions.*"

If the co-ordinates are changed from the axis X to the axis Y, the same rule obtains with respect to the fluxional coefficients $\frac{d^2x}{dy^2}$, $\frac{d^3x}{dy^3}$. . . .

Since at an inflexion the tangent has contact of the second order, it intersects the curve.

In this investigation the value of p has not been considered; it only marks the position of the tangent; when $p = 0$, it shows that the tangent at the inflexion is parallel to the axis.

If p is infinite, or the tangent is perpendicular to the axis, all the succeeding fluxional coefficients will be infinite at the same time, and the proposition on which the investigation is founded, viz. (3. 3) fails. In order to ascertain whether

there is an inflexion in this case, recourse must be had to the following method.

7. *Required to find whether there is an inflexion at a proposed point of a known curve.*

Let a be the value of the abscissa at the proposed point; calculate the value of q in terms of x from the curve's equation; substitute in the result $a + h$, $a - h$ for x; then there is an inflexion or not according as the signs of (q) are different or the same, when h is indefinitely diminished (Art. 2.)

Cor. At an inflexion the radius of curvature becomes infinite, and the curvature vanishes.

8. *Required to investigate the points of contrary flexure of a spiral.*

If the polar equation be given, deduce from it the equation between the rectangular co-ordinates; and apply the rule of Art. 6.

Or thus. The radius vector increasing, the form of a spiral is such, that when it is *concave* to the pole, the perpendicular on the tangent necessarily increases, and when it is *convex*, the perpendicular diminishes. At an inflexion, therefore, the perpendicular on the tangent is either a maximum or a minimum, and it may be found from the equation between p and r.

9. *Examples.*

Ex. 1. $y = b + ax^2$.

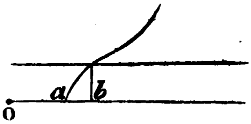
$p = 2ax$
$q = 2a$

which shows that there is no inflexion, and that the curve is always *convex* to the axis; for the curve does not cut the axis.

Ex. 2. $y = b + (x - a)^3$.

$p = 3.(x - a)^2$
$q = 2.3(x - a)$
$r = 6.$

There is an inflexion at $x=a, y=b$; also the tangent of the inflexion is parallel to the axis; and the form of the curve is that of the annexed figure.



Ex. 3. $a^3y = x^4$.

$$p = \frac{4.x^3}{a^3}, \quad q = \frac{3.4x^2}{a^3}, \quad r = \frac{2.3.4x}{a^3}, \quad s = \frac{2.3.4}{a^3}$$

$x = 0, y = 0$ does not indicate an inflexion, since the first fluxional coefficient which does not vanish is of an *even* order. There is, however, an inflexion at the origin, as will be shown in the course of the chapter; but it is *invisible*.

Ex. 4. $y = ax + bx^2 - cx^3$.

$$p = a + 2bx - 3cx^2, \quad q = 2b - 2.3cx, \quad r = -6c$$

$x = \frac{b}{3c}$ gives an inflexion: the curve is convex before, and concave after the value $x = \frac{b}{3c}$.

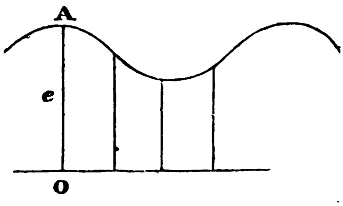
Ex. 5. $y = e^{\cos.x}$.

$$p = -e^{\cos.x}\sin.x, \quad q = e^{\cos.x}(\sin.^2 x - \cos.x)$$

If there is an inflexion, $\sin.^2 x - \cos.x = 0$, or $\cos.x = -\frac{1 \pm \sqrt{5}}{2}$;

and to find the sign of (r), we have by the rule of (6. 9), sign of (r) = sign of $e^{\cos.x}(2\sin.x\cos.x + \sin.x)$, which does not vanish when $\cos.x = \sin.^2 x$.

This curve has two inflexions at the points $x = \cos.^{-1}\frac{-1+\sqrt{5}}{2}$, $y = e^{\frac{-1+\sqrt{5}}{2}}$ and $x = \cos.^{-1}\frac{-1-\sqrt{5}}{2}, \ldots$ $y = e^{-\frac{1+\sqrt{5}}{2}}$. Its form is that of the annexed figure.



Ex. 6. $a^2y^2 + x^2y^2 = a^4$; or $y = \frac{a^2}{\sqrt{a^2+x^2}}$.

$$p = -\frac{a^2x}{(a^2+x^2)^{\frac{3}{2}}}$$

$$q = -\frac{a^3}{(a^2+x^2)^{\frac{3}{2}}} + \frac{3a^2x^2}{(a^2+x^2)^{\frac{5}{2}}}$$

$$= \frac{2a^2x^2 - a^4}{(a^2+x^2)^{\frac{5}{2}}}$$

There is an inflexion at $x = \pm\frac{a}{\sqrt{2}}$, $y = \pm a\sqrt{\frac{2}{3}}$.

The curve is symmetrical on both the axes; and is of the form of a double conchoid, having four inflexions.

Ex. 7. $y = \text{v.s.}^{-1}x$.

$x = \text{v.s.}y = 1 - \cos.y$ $\therefore$

$$\frac{dx}{dy} = \sin.y$$

$$\frac{d^2x}{dy^2} = \cos.y$$

$$\frac{d^3x}{dy^3} = -\sin.y$$

There is an inflexion at $y = \frac{\pi r}{2}$, $x = r$; and since $\cos.\frac{3\pi r}{2}, = 0$, $\cos.\frac{5\pi r}{2} = 0\ldots$ there are inflexions at $x = r$ and $y = \frac{\pi r}{2},\ldots$ $\frac{3\pi r}{2}\ldots\frac{(2n+1)\pi r}{2}$, where n may be any integer.

Ex. 8. $y = \tan.\frac{1}{2}\text{v.s.}^{-1}x$. (Vid. 9. 4. Ex. 14.)

First to clear the equation of circular functions; substitute $2z = \text{v.s.}^{-1}x$ $\therefore$ $x = \text{v.s.}2z = 1 - \cos.2z = 2\sin.^2z$. Also

$$y = \tan.z = \frac{\sin.z}{\cos.z} = \frac{\sin.z}{\sqrt{1-\sin.^2z}} = \left(\frac{x}{2-x}\right)^{\frac{1}{2}} \therefore \text{ we have}$$

$$p = \frac{1}{2}\left(\frac{2-x}{x}\right)^{\frac{1}{2}}\left\{\frac{1}{2-x} + \frac{x}{(2-x)^2}\right\}$$

$$= \frac{x}{(2x-x^2)^{\frac{3}{2}}}$$

$$q = \frac{x(2x-1)}{(2x-x^2)^{\frac{5}{2}}}$$

$q = 0$ when $2x - 1 = 0$, or $x = \frac{1}{2}$; and if $\frac{1}{2} \pm h$ be substituted for x, the values of q are the one positive and the other negative; or the curve is concave before, and convex after $x = \frac{1}{2}$.

When $x = 2$, $p = \infty$, and $q = \infty$; and to determine whether this indicates an inflexion, substitute $2 \pm h$ for x, of which the first makes q impossible; or there is not an *inflexion* at $x = 2$.

Ex. 9. A spiral in which the radius vector varies as the nth power of the angle described.

(10. 12. Ex. 3. (3.)) $p = \dfrac{r^{\frac{n+1}{n}}}{\sqrt{r^{\frac{2}{n}} + n^2 b^{\frac{2}{n}}}}$, which is to be a maximum, or $\dfrac{n+1}{n} lr - \dfrac{1}{2} l(r^{\frac{2}{n}} + n^2 b^{\frac{2}{n}}) \ldots$

$= \text{maximum}; \therefore \dfrac{n+1}{n} \dfrac{1}{r} = \dfrac{1}{n} \dfrac{r^{\frac{2}{n}-1}}{r^{\frac{2}{n}} + n^2 b^{\frac{2}{n}}} \therefore \ldots$

$(n+1)(r^{\frac{2}{n}} + n^2 b^{\frac{2}{n}}) = r^{\frac{2}{n}} \therefore r = b\{-n(n+1)\}^{\frac{n}{2}}.$

Hence, there is an inflexion only when n is a negative fraction less than unity.

In the *Lituus* $n = -\frac{1}{2} \therefore r = b\sqrt{2} = 2a \therefore \theta = \frac{1}{2}$.

Ex. 10. $y^3 = x^5$, or $y = x^{\frac{5}{3}}$.

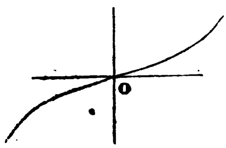
$p = \dfrac{5}{3} x^{\frac{2}{3}}$

$q = \dfrac{2}{3} \cdot \dfrac{5}{3} x^{-\frac{1}{3}}$

$r = -\dfrac{1}{3} \cdot \dfrac{2}{3} \cdot \dfrac{5}{3} x^{-\frac{4}{3}}$

At the origin $q = \infty$; and to determine whether there is an inflexion, for x substitute $+h$ and $-h$, and the resulting values of q are $\dfrac{2}{3} \cdot \dfrac{5}{3} \dfrac{1}{h^{\frac{1}{3}}}$, and ...

$-\dfrac{2}{3} \cdot \dfrac{5}{3} \dfrac{1}{h^{\frac{1}{3}}}$; or the curve is convex on each side of the axis (Art. 3); and it passes through o; which must therefore be an inflexion.

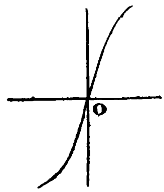


Ex. 11. $y^3 = x$, or $y = x^{\frac{1}{3}}$.

$p = \dfrac{1}{3} x^{-\frac{2}{3}}$

$q = -\dfrac{1}{3} \cdot \dfrac{2}{3} x^{-\frac{5}{3}}$

There is an inflexion at o of the form of the annexed figure.



10. PRAXIS.

1. $x^3 + ax^2 + b^2y = 0 \therefore x = -\frac{a}{3}$.

2. $y = x^4 - 12x^3 + 48x^2 - 64x$. The curve is convex when x is less than 2; concave when x is between 2 and 4; and afterwards convex.

3. $a^4y = 3x^5 - 35ax^4 + 140a^2x^3 - 240a^3x^2$. The curve is concave before $x = a$; convex between $x = a$, $x = 2a$; concave between $x = 2a$, $x = 4a$; and afterwards convex.

4. $y = b + (x - a)^n$. There is or is not an inflexion at $x = a$ according as n is odd or even where n is an integer greater than unity.

5. $y = \frac{a}{x}\sqrt{ax - x^2}$ (8. 7. Ex. 5.) $\therefore x = \frac{a}{3}, y = \pm\frac{a}{\sqrt{3}}$.

6. $ay^2 - xy^2 - a^2x = 0 \therefore x = \frac{a}{4}, y = \pm\frac{a}{\sqrt{3}}$.

7. $y = x + 36x^2 - 2x^3 - x^4$. This curve passes through the origin inclined at $\angle$ 45° without inflexion. There are inflexions at $x = 2$ and $x = -3$.

8. $y = x + \cos.x - 1 \therefore x = \frac{\pi}{2}, \frac{3\pi}{2} \dots \frac{(2n+1)\pi}{2}$,

and $y = x - 1$.

9. $x = ly^2 \therefore x = 8, y = e^2$.

10. The line of sines. The equation is $y = \sin.x$; and the curve cuts the axis in the origin and other points at $\angle$ 45, without inflexion.

11. $y = \text{v.s.}\,2x \therefore y = r$, & $x = \frac{\pi r}{2}, \frac{3\pi r}{2} \dots \frac{(2n+1)\pi r}{2}$.

12. A spiral whose equation is $r = \frac{a\theta^2}{\theta^2 - 1}$ (10.12. Ex.11.) $r = \frac{3a}{2}$, $\theta = \sqrt{3}$; and θ may be calculated in degrees as in Ch. 2. Art. 18. Cor.

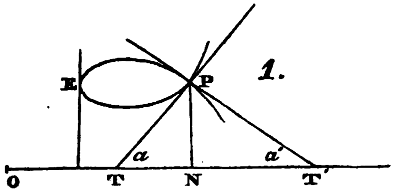
For additional examples, vid. Ch. 7. Praxis to Art. 17.

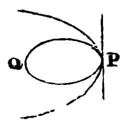
11. *Multiple Points.*

Def. 1. A *Multiple point* is one in which two or more branches of the curve cut or touch each other. If the branches cut, the point is of the first species; and of the second, if they touch. They are represented in fig. 1 and fig. 2.

They are *double* or *triple* points according to the number of the intersecting branches.

The two species of multiple points have this property in common, that at these points two or more co-ordinates of the curve coincide; and they differ in that a point of the second species has only one tangent.





Def. 2. The point E of fig. 1, which is called the *limit* of the curve in the direction of the axis ON, is a singular point. It has one property in common with multiple points; for two ordinates of the curve coincide at E; but it may be distinguished from them by the circumstance, that if the axis be changed it ceases to possess any singular property.

Def. 3. *Cusps* may be considered as multiple points of the second species. At a cusp the curvature is not only discontinued as in a point of contrary flexure, but the curve also suddenly changes its direction. The French mathematicians call them *rebroussemens.* They are of two species; in the first, the branches of the curve lie on opposite sides of their tangent; and in the second, on the same side.

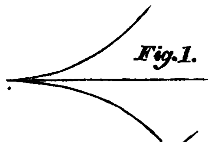


Fig. 1.


Fig. 2.

12. *If a curve's equation be free from radicals, the value of* $\frac{dy}{dx}$ *at a multiple point takes the form of* $\frac{0}{0}$.

For $\frac{dy}{dx}$ marks the inclination of the tangent; and since at a multiple point of the first species there is more than one tangent, it follows that $\frac{dy}{dx}$ must assume the indeterminate form of $\frac{0}{0}$, unless there are radicals in the equation; in

which case $\frac{dy}{dx}$ indicates as many branches as the radical has real values.

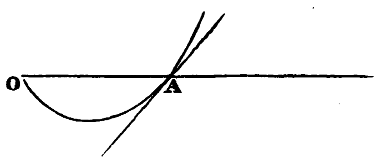
When $\frac{dy}{dx} = \frac{0}{0}$, its values must be found as in Ch. 5. 21.; and their number will give the degree of multiplicity of the proposed point. If $\frac{dy}{dx}$ has two equal values, it indicates either a limit of the curve or a double point of the second species.

Otherwise. Let α and α' be the tangents of the $\angle^s$ PTN, PT'N; differentiate the curve's equation $F(x, y) = 0$, and let the result be $P dx + Q dy = 0$, or $\frac{dy}{dx} = -\frac{P}{Q}$.

Now $P + Q.\frac{dy}{dx} = 0$; therefore $P + Q\alpha = 0$. Similarly $P + Q\alpha' = 0$; wherefore $Q(\alpha - \alpha') = 0$; but α and α' are different quantities, therefore $Q = 0$, and consequently $P = 0$; and $\frac{dy}{dx}$, which $= -\frac{P}{Q}, = \frac{0}{0}$.

This proposition is not true *e converso*.

For take the example $y^3 - x(x-1)^3 = 0$; from which $p = \frac{(x-1)^3 - 3x(x-1)^2}{3y^2} = \frac{0}{0}$ when $x = 1$. But if the curve be traced, it will be seen that it can have but one branch which cuts the axis at $\angle$ 45°.



13. *Required to investigate the number of the multiple points of a curve, and the degree of their multiplicity.*

Let $y^n + Py^{n-1} + Qy^{n-2} + \ldots + Ty + U = 0 = u$ be the curve's equation free from radicals, where P, Q, ... T, U are explicit functions of x. If a determinate value a be assigned to x, this equation has n roots (Alg. 518); and if a belong to a multiple point where m branches intersect, m of these roots are equal; let b be the equal root; then, since $u = 0$ has m equal roots, its limiting equation $\frac{du}{dy} = 0$

contains $m-1$ roots $= b$ (Alg. 319). Find then all the values of x and y which satisfy both the equations $u = 0$, $\frac{du}{dy} = 0$; let their values be a, b; a', b'; a'', b'', &c.; substitute them in the expression for p; then all the resulting expressions which belong to a multiple point take the form of $\frac{0}{0}$, and their real value may be obtained by the method of Ch. 5. Art. 21. If p has more than one real value there is a multiple point, and the degree of its multiplicity depends upon the number of p's real values.

At a multiple point the limiting equation with respect to x, or $\frac{du}{dx} = 0$ must also contain $m-1$ roots $= b$; hence the required co-ordinates must be such as to satisfy the three equations $u=0, \frac{du}{dy} = 0, \frac{du}{dx} = 0$.

In order then to determine the number of multiple points, we may find all the values of x and y which satisfy the equations $u = 0, \frac{du}{dy} = 0, \frac{du}{dx} = 0$; if there are any pairs which satisfy all three, they belong to a multiple point, and the corresponding values of p, which will take the form of $\frac{0}{0}$, will indicate as before its degree of multiplicity.

Those pairs which satisfy the original equation and only one of the limiting equations, in which case p does not take the form of $\frac{0}{0}$, belong to a limit in the direction of one of the axes.

If p has no real values, the point is a conjugate point. (Vid. 7. 17. Ex. 4. Def.)

14. *Examples.*

Ex. 1. $ay^2 + x^3 - bx^2 = 0 = u$.

$$\left.\begin{aligned} \frac{du}{dy} &= 2ay = 0 \\ \frac{du}{dx} &= x(3x-2b) = 0 \end{aligned}\right\} \begin{aligned} &\therefore x = 0, y = 0 \text{ is the only possible} \\ &\text{multiple point; and since } \ldots\ldots \\ &p = \frac{2bx - 3x^2}{2ay} \quad \therefore \text{ (5. 21. Ex. 1.)} \end{aligned}$$

$(p)^2 = \frac{b}{a}$; or there is a double point at the origin.

Ex. 2. $y^3 - x(x-1)^3 = 0 = u$.

$$\left.\begin{aligned}\frac{du}{dy} &= 3y^2 = 0\\ \frac{du}{dx} &= -(x-1)^3 - 3x(x-1)^2 = 0\end{aligned}\right\}\therefore y = 0,\ x = 1 \text{ is the only possible multiple point; and}$$

$$p = \frac{(x-1)^3+3x(x-1)^2}{3y^2} = \frac{0}{0} = \frac{(x-1)^2+x(x-1)}{yp}\ \ldots$$

$= \dfrac{3(x-1)+x}{p^2}$ $\therefore p^3 + 3 = 0$, which contains only one real root; or $y = 0$, $x = 1$ is not a multiple point.

Ex. 3. $y = \dfrac{\pm(ax - x^2)}{\sqrt{2ax - x^2}}$.

Clear the equation of radicals, then $y^2(2ax - x^2) = (ax - x^2)^2$; divide by x that the results may not indicate a greater degree of multiplicity than the points possess, and we have

$$\left.\begin{aligned}\frac{du}{dy} &= 2y(2a-x) = 0\\ \frac{du}{dx} &= -y^2-(a-x)(a-3x) = 0\end{aligned}\right\}\text{and } p = \frac{y^2+(a-x)(a-3x)}{2y(2a-x)}.$$

The only multiple point is at $x = a$, $y = 0$; and

$$p = \frac{yp-2a+3x}{p(2a-x)-y} = \frac{1}{p}\ \therefore p = \pm 1.$$

The curve is of the form of the lower conchoid, the branches intersecting the axis at $\angle\ 45^\circ$.

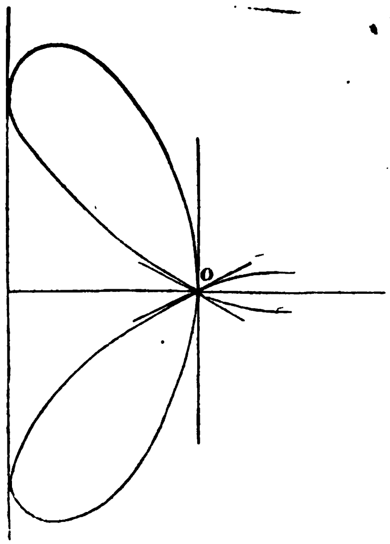
Ex. 4. $y^4 + 2axy^2 - ax^3 = 0 = u$.

$$\left.\begin{aligned}\frac{du}{dy} &= 4y(y^2 + ax) = 0\\ \frac{du}{dx} &= a(2y^2 - 3x^2) = 0\end{aligned}\right\}\text{and } p = \frac{3ax^2-2ay^2}{4y^3+4axy}.$$

$x = 0$, $y = 0$ is the only pair which satisfies all three, and

$$p = \frac{3ax-2ayp}{6y^2p+2ay+2axp} = \frac{3a-2ap^2}{12yp^2+4ap}\ \therefore\ 12yp^3 + 6ap^2 -$$

$3a = 0$; in which equation, if p be finite $6ap^2 - 3a = 0$, or $p = \pm\frac{1}{\sqrt{2}}$. The third value of p is infinite; whence.... there is a triple point at the origin, one branch cutting the axis at right angles, and the two others at angles whose tangents are $\pm\frac{1}{\sqrt{2}}$.



Also $x = -a$, $y = \pm a$ satisfy $u = 0$ and $\frac{du}{dy} = 0$, but not $\frac{du}{dx} = 0$ and $(p) = \infty$; $\therefore$ they indicate limits in the direction x. The form of the curve is that of the annexed figure. (Garnier, Calc. Diff. p. 297.)

Ex. 5. $y^4 - x^5 + x^4 + 3x^2y^2 = 0 = u$.

$$\left.\begin{aligned}\frac{du}{dy} &= 2y(2y^2 + 3x^2) = 0\\ \frac{du}{dx} &= -x(5x^3 - 4x^2 - 6y^2) = 0\end{aligned}\right\}\text{and } p = \frac{5x^4 - 4x^3 - 6xy^2}{4y^3 + 6x^2y}.$$

The only possible multiple point is $x = 0, y = 0$; and

$$p = \frac{10x^3 - 6x - 3y^2 - 6xyp}{6y^2p + 6xy + 3x^2p} = \frac{10x^2 - 4x - 4yp - 2xp^2}{4yp^2 + 2y + 4xp} \dots\dots$$

$$= \frac{10x - 2 - 3p^2}{2p^3 + 3p} \therefore p^4 + 3p^2 + 1 = 0;$$ and since all the roots of this biquadratick are impossible, there is a conjugate point at the origin.

Ex. 6. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

$$\left.\begin{array}{l}\dfrac{du}{dy} = y\left\{2(x^2+y^2)+a^2\right\} = 0 \\ \dfrac{du}{dx} = x\left\{2(x^2+y^2)-a^2\right\} = 0\end{array}\right\} \& \; p = \frac{a^2x - 2x(x^2+y^2)}{a^2y+2y(x^2+y^2)}.$$

The only multiple point is $x = 0, y = 0$; and $p = \dfrac{a^2}{a^2p}$ $\therefore p^2 = 1$; or there is a double point at the origin, the branches cutting the axis at $\angle$ 45°.

Also $\pm y = 0$, $\pm x = a$ satisfy $u = 0$ and $\dfrac{du}{dy} = 0$, but not $\dfrac{du}{dx} = 0$; and consequently they belong to a limit in the direction x. The limits in the direction Y are . . . .

$$x = \pm\frac{a}{2}\sqrt{\frac{3}{2}}, y = \pm\frac{a}{2}\sqrt{\frac{1}{2}}.$$

Ex. 7. $x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0.$

$$\left.\begin{array}{l}\dfrac{du}{dy} = - 6ay(y+a) = 0 \\ \dfrac{du}{dx} = 4x(x^2-a^2) = 0\end{array}\right\} \text{and } p = \frac{2x(x^2-a^2)}{3ay(y+a)}.$$

$\left.\begin{array}{l}y = 0 \\ x = \pm a\end{array}\right\}$ and $\left.\begin{array}{l}y = -a \\ x = 0\end{array}\right\}$ are all the possible multiple points.

$$\text{At } y = 0, x = \pm a, p = \frac{6x^2-2a^2}{6ayp+3a^2p} = \frac{4a^2}{3a^2p} \therefore p^2 = \frac{4}{3}.$$

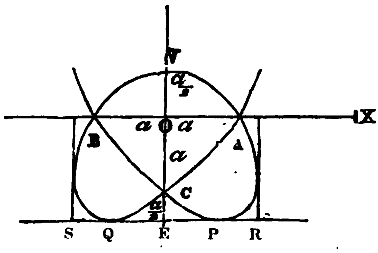
$$\text{At } y = -a, x = 0, p = \frac{-2a^2}{-3a^2p} \therefore p^2 = \frac{2}{3}.$$

Hence there are three double points, of which the branches of the two first are inclined to the axis at $\tan.^{-1} \pm \dfrac{2}{\sqrt{3}}$; and the branches of the third at . . . . . $\tan.^{-1} \pm \sqrt{\dfrac{2}{3}}$.

And to find the limits in the directions x and Y, there are

$\left.\begin{array}{l} x = 0 \\ y = \frac{a}{2} \end{array}\right\} \left.\begin{array}{l} x = \pm a \\ y = -\frac{3a}{2} \end{array}\right\}$ which satisfy $u = 0$ and $\frac{du}{dx} = 0$, but not $\frac{du}{dy} = 0$:

also $\left.\begin{array}{l} x = \pm a\sqrt{2} \\ y = -a \end{array}\right\}$ satisfy $u = 0$ and $\frac{du}{dy} = 0$, but not .... $\frac{du}{dx} = 0$; consequently there are three limits in the direction Y, and two in the direction X, of which the tangents of the former are parallel; and of the latter perpendicular to the axis.



In the annexed diagram OA = OB = OC = a, OV $= \frac{a}{2}$, CE $= \frac{a}{2}$; EP = EQ $= a$; ER = ES $= a\sqrt{2}$. (Garnier, Calc. Diff. p. 296.)

Having determined by the above method the number and position of the multiple points, it remains to ascertain the nature of each; or the situation of the branches which issue from it.

If the fluxional coefficients present no singularity, we may be assured that the proposed point is not a singular point; but the reverse of this proposition is not true. Whenever then p, q, &c. become either 0, ∞, or take the form of $\frac{0}{0}$, it becomes a problem of some importance in the discussion of a curve to ascertain the situation of the branches in the neighbourhood of the proposed point.

15. *Required to investigate the nature of any proposed point of a given curve.*

Transfer the origin to the proposed point; and develope y, which is the same as $y' - y$, or k in the original equation, by the theorems of Ch. 4 in a series of the form $Ax^{\alpha} + Bx^{\beta} + Cx^{\gamma} +$ &c. where α, β, γ, &c. are increasing positive magnitudes.

Diminish x and y indefinitely; then (3. 3) the nature of

the branch issuing from the proposed point depends upon the equation $y = \mathrm{A}x^{\alpha}$.

I. Let $\alpha = 1$; then $y = \mathrm{A}x$, and the curve crosses the axis at $\tan.^{-1}\mathrm{A}$.

II. Let α be greater than 1; then we have

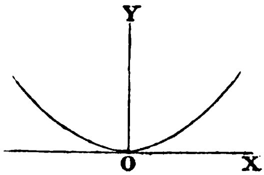
$$\left.\begin{array}{l} y = \mathrm{A}x^{\alpha} \\ p = \alpha \mathrm{A}x^{\alpha-1} \\ q = \alpha(\alpha - 1)\mathrm{A}x^{\alpha-2} \end{array}\right\}.$$

This resolves itself into three cases.

Case 1. Let $\alpha = 2i$, or $= \dfrac{2i}{2i'+1}$.

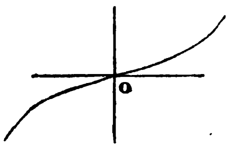
Here the curve cannot cross the axis, for y cannot be negative; it passes through the origin touching the axis; and it is convex.

The origin is not a singular point; for maxima and minima are not usually considered as such.



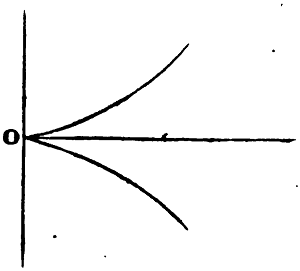
Case 2. $\alpha = 2i + 1$, or $= \dfrac{2i+1}{2i'+1}$.

Here the curve crosses the axis, touching it; and both branches are convex; and there is an inflexion at O.



Case 3. $\alpha = \dfrac{2i+1}{2i'}$.

Here the curve cannot cross the axis Y; and since y has two values, the one positive and the other negative, the two branches are on opposite sides of X, touching it. Also since y and q have the same sign, both the branches are convex; and the origin is a cusp of the first species.



III. Let α be less than 1.

These cases are all reducible to the preceding; for since $y = \mathrm{A}x^{\alpha}$, $x = \mathrm{A}'y^{\frac{1}{\alpha}}$; from which the position of the curve with respect to the axis Y may be determined.

16. It appears from the preceding article that the equation $y = \text{A}x^{\alpha}$ cannot indicate a cusp of the second species; for there is no value of α which causes the values of y to be both positive or both negative. If then such points have existence, they must arise from some property of those terms of the series which were neglected in the investigation. Let us then proceed to inquire whether there is any form of y's developement which can indicate a cusp of the second species.

Now suppose, α being greater than 1, that one of the exponents β, γ, &c. is of the form $\frac{2i+1}{2i'}$; let $\text{B}x^{\beta}$ be the term; then $\text{B}x^{\beta}$, though it may be neglected with respect to magnitude when compared with $\text{A}x^{\alpha}$, yet it implies a restriction which may not be indicated by $\text{A}x^{\alpha}$; for it shows that the curve cannot cross the axis Y, whatever be the value of α.

When this obtains, the origin is a cusp of the second species. Thus let the curve's equation be . . . .

$y = \text{A}x^2 \pm \text{B}x^{\frac{5}{2}}$.

Here $(p) = 0$, or the branches touch the axis X. Also the curve cannot cross the axis Y, for x cannot be negative; nor can y be negative, since x may be assumed so small that $\text{A}x^2$ may be always greater than $\text{B}x^{\frac{5}{2}}$: and q is positive for both the branches; consequently the origin is a cusp of the second species, the two branches being convex to the axis X.

If $\alpha = 1$, there is likewise a cusp of the second species at the origin, unless the term in which the exponent $\frac{2i+1}{2i'}$ occurs should be the *second* term of the developement; in which case, there is a cusp of the first species. The student may assure himself of this by the examples $y = \text{A}x + \text{B}x^{\frac{3}{2}}$ and $y = \text{A}x + \text{B}x^2 + \text{C}x^{\frac{5}{2}}$; they both indicate cusps at the origin; the one of the first species, the other of the second; the tangents of both are inclined to the axis at $\tan.^{-1}\text{A}$; and the branches of the second curve are convex.

Hence it appears that there is a cusp either of the first or

second species when, α being greater than or $= 1$, any of the exponents of the developement is of the form $\frac{2i+1}{2i'}$.

If α is less than 1, as in the example $y = x^{\frac{1}{3}} \pm x^{\frac{3}{2}}$; in order to find x in terms of y, we have $x^{\frac{1}{3}} = y \mp x^{\frac{3}{2}}$, or $x = y^3 \mp 3y^2x^{\frac{3}{2}} + 3yx^3 \mp x^{\frac{9}{2}} =$ (Lagrange's theor. 4. 41.) $y^3 \mp 3y^{\frac{7}{2}} + 3y^4 \mp$ &c., which indicates a cusp of the second species which touches and is convex to the axis Y.

17. If the developement contains more than one exponent of the form $\frac{2i+1}{2i'}$, which are independent of each other, y has more than two values; and if n be the number of such terms, the number of cusps shall be $(n-1)^{n-1}$.

18. When the third term of the developement is the first which has an exponent of the form $\frac{2i+1}{2i'}$, the branches of the cusp have contact of the second order, and a curve whose equation is $y = \text{A}x^{\alpha} + \text{B}x^{\beta}$ drawn through the cusp will pass between its branches. If the $(n+1)$th term is the first whose exponent is $\frac{2i+1}{2i'}$, the contact of the branches is of the nth order.

19. If any of the coefficients contain $\sqrt{-1}$, the point is a conjugate point. There is one exception; suppose $\text{B}x^{\beta}$ to be the term in which B contains $\sqrt{-1}$; then if $\beta = \frac{2i+1}{2i'}$, x^{β} when x is changed to $-x$ also contains $\sqrt{-1}$; and the form of B may be such that the product shall be a possible negative quantity. In this case the curve is in the second quadrant; but as no part of this branch of the curve can now be in the first and fourth quadrants (Ex. hyp.), it does not cross the axis, but forms a cusp the same as before, except that it is situated in the second quadrant.

If $\text{B} = a + b\sqrt{-1}$, the point must be a conjugate point whatever be the form of β.

Throughout the whole of the investigation in this and the preceding articles, we have supposed A to be positive; if it be negative, this will only change the quadrants in which the branches are situated; the figure in other respects will be the same.

If the sign of B be changed, this produces no alteration in the figure except when $\alpha = 1$; in which case the convex curves become concave, and vice versâ.

20. *Examples.*

Ex. 1. $ay^2 = x^3$, the semi-cubical parabola.

$y = \dfrac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$ ∴ (14. Case 3.) the origin is a cusp of the first species whose branches touch the axis. Vid. fig. Ch. 11. 14. Ex. 1.

Ex. 2. $a(y-b)^2 = (x-c)^3$ is the same curve as the former example, a semi-cubical parabola, whose vertex is at $x = c$, $y = b$; and whose axis is parallel to x.

Ex. 3. $a^2y^3 = x^5$.

$(p) = 0$ $(q) = \infty$	The curve touches the axis; the origin is an inflexion; and the curve is convex.

Ex. 4. $a^2x = y^3$.

The origin is an inflexion; and it is convex to and touches the axis Y.

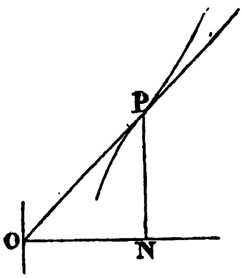
Ex. 5. $a^7(y-x)^2 = x^9$, or $y = x + \dfrac{x^{\frac{9}{2}}}{a^{\frac{7}{2}}}$.

$p = 1$ $q = \dfrac{7}{2}\,\dfrac{9}{2}\,\dfrac{x^{\frac{5}{2}}}{a^{\frac{7}{2}}}$	There is a cusp of the first kind at the origin; its tangent being inclined to the axis at ∠ 45°.

Ex. 6. $y - x = (x-a)^{\frac{5}{3}}$.

Transfer the origin to the point $x = a$, then the equation becomes $y-(x+a) = x^{\frac{5}{3}}$, or $y - a = x + x^{\frac{5}{3}}$; transfer this again to $y = a$, and the equation is $y = x + x^{\frac{5}{3}}$.

From this equation $p = 1$; and y and q have the same sign, either both positive or both negative; hence there is an inflexion at $x = a$, $y = a$, whose tangent passes through the origin inclined to the axis at $\angle\ 45^\circ$, as represented in the figure.



Ex. 7. $y - b = (x-a)^{\frac{1}{3}} + (x-a)^{\frac{3}{4}}$.

Transfer the origin to $x = a$, $y = b$; and the equation is $y = x^{\frac{1}{3}} + x^{\frac{3}{4}}$, and the nature of the point may be found as in Art. 15, when α is less than 1.

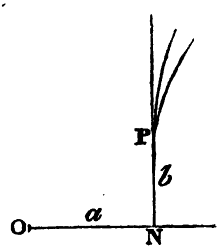
To find the nature of the point independent of the rules of Art. 15, we have

$$p = \frac{1}{3}x^{-\frac{2}{3}} + \frac{3}{4}x^{-\frac{1}{4}}$$

$$q = \frac{1}{9}x^{-\frac{4}{3}} - \frac{3}{16}x^{-\frac{5}{4}}$$

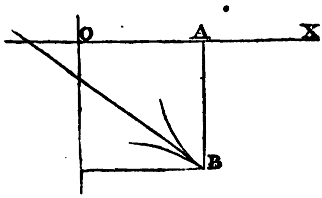
Hence $(p) = \infty$ ∴ x cannot be negative; (y) has two values $x^{\frac{1}{3}} \pm x^{\frac{3}{4}}$, neither of which is negative, since $x^{\frac{3}{4}}$ is less than $x^{\frac{1}{3}}$; (q) is in both cases positive.

There is then at $x = a$, $y = b$ a cusp of the second species, whose tangent is at right angles to the axis of the curve.



Ex. 8. $(2y + x + a)^2 = 2(a-x)^5$.

When $x = a$, $y = -a$, and $p = \frac{0}{0}$; transferring the origin to A, and reckoning the abscissa in the direction AO, the equation becomes $(2y + 2a - x)^2$



$= 2x^5$; transfer the origin again to $y = -a$, and the equation is $(2y-x)^2 = 2x^5$, or $y = \frac{x}{2} \pm \frac{x^{\frac{5}{2}}}{\sqrt{2}}$.

Hence there is a cusp of the first species as represented in the figure; its tangent is inclined to the axis passing through B and to the axis OA at $\tan.^{-1}\frac{1}{2}$.

Ex. 9. $y - b = (x^2 - a^2)^{\frac{3}{2}}$.

The equation, when the origin is transferred to $x = a$, $y = b$, is $y = (2ax + x^2)^{\frac{3}{2}} = (2ax)^{\frac{3}{2}}\left(1 + \frac{x}{2a}\right)^{\frac{3}{2}} = (2ax)^{\frac{3}{2}} + \frac{3a^{\frac{1}{2}}x^{\frac{5}{2}}}{2^{\frac{1}{2}}} + \&c.$

Hence there are two cusps of the first species at $x = \pm a$, $y = b$, whose axes are parallel to the axis of the curve.

The curve is symmetrical on the axis y. There are no conjugate points.

21. In this article we shall recapitulate summarily the different circumstances indicated by the fluxional coefficients.

(1.) Let p be finite, and let it have but one value; then it marks the position of the tangent; and it cannot indicate any other circumstance, unless the value of q be also taken into consideration.

(2.) If $p = 0$, the tangent is parallel to the axis; and unless the point is an inflexion or a cusp, there is a maximum or minimum; and p changes its sign in passing through zero.

(3.) If $p = \infty$, the tangent is at right angles to the axis; and unless there is an inflexion, the point is a cusp whose tangent is perpendicular to the axis. The cusp in this case may be considered as a maximum or minimum where p changes its sign in passing through infinity.

(4.) If p, or any of the fluxional coefficients contain $\sqrt{-1}$, this shows that the point, if it belong to the curve, is a conjugate point.

(5.) If p take the form of $\frac{0}{0}$, it may be a conjugate or a

multiple point, and its degree of multiplicity may be found; but not its nature, without calculating q.

(6.) Let q be finite; and let it have but one value; then the only circumstance which it indicates, is whether the curve is convex or concave.

(7.) If $q = 0$, the point may be a maximum or minimum, or an inflexion, or a cusp. To ascertain whether there is a cusp, the contiguous values of the ordinate must be developed in series ascending by the powers of the increment of the abscissa; and if the resulting series do not indicate a cusp, there is an inflexion, if r is not $= 0$; and a maximum or minimum, if $p = 0$, $r = 0$, and s is not $= 0$.

There is this difference between a cusp of the first and of the second species; the first shows itself within the two first terms of the developement; but of the second, we cannot be assured of its non-existence without ascertaining that there is not any term of the developement whose exponent $= \frac{2i+1}{2i'}$.

(8.) If $q = \infty$, there is either an inflexion or a cusp. In these cases q changes its sign in passing through infinity, except when the point is a cusp of the first species whose tangent is perpendicular to the axis.

(9.) If q takes the form of $\frac{0}{0}$, this shows that more than one branch may belong to the point (11. 2); and calculating q by the method of Ch. 5. 21; if it possesses two real unequal values, there is a double point of the second species, whose branches have contact of the first order, and the position of whose tangent depends upon the value of p.

If the two values of q are equal, the branches have contact of the second order, and they intersect.

Generally, "if any of the coefficients, the $n + 1$th for instance, takes the form of $\frac{0}{0}$, there is a multiple point of the second species; its degree of multiplicity depends upon the number of the real values which the coefficient possesses; and the order of contact of all the branches is the nth, if the values are all unequal."

22. When y is an explicit function of x, the process of finding the number of the multiple points and their nature becomes more simple by observing that they arise from assigning such a value to the variable as causes the *coefficient* of the radical in fx to $= 0$; which is the case in which the fluxional coefficients do not fail. (Vid. Ch. 5. Art. 2.) The exponent of the radical indicates the number of the intersecting branches; and the exponent of the coefficient which $= 0$, indicates the nature of the point of intersection.

If the same value of the variable causes both the coefficient and the radical to $= 0$, the point may be a multiple point.

Ex. 1. Thus in Art. 13., Ex. 1., $x = a$ causes the coefficient of the radical to disappear; and since it reappears in $f'x$, it indicates a double point.

In the same example, $x = 0$ causes both the coefficient and its radical to $= 0$, and it belongs only to a limit.

Ex. 2. $y = (x-a)\sqrt{x-b} + c$ where a is greater than b.

$x = a$, $y = c$ indicates a double point, as will be seen from clearing the equation of radicals and proceeding as in Art. 3.

If $a = b$; $x = a$, $y = c$ is a conjugate point.

Ex. 3. $y = (x-a)^2\sqrt{x-b}+c$.

$x = a$, $y = c$ is a double point of the second species; and the branches touch with contact of the first order.

Ex. 4. $y = (x-a)^3\sqrt{x-b}+c$.

$x = a$, $y = c$ is a double point of the second species; and the branches touch with contact of the second order.

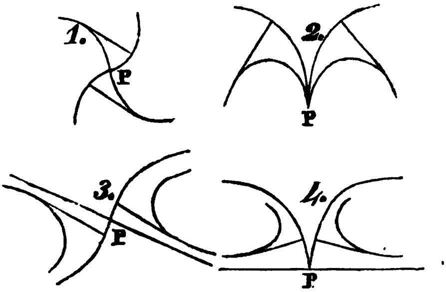
Ex. 5. $y^3-(x-1)^3x = 0$.

$y = (x-1)x^{\frac{1}{3}}$; the exponent of the radical indicates only one branch, and consequently there is no multiple point at $x = 1$.

23. *Required the nature of that point of the evolute which corresponds to a point of contrary flexure, or to a cusp in the involute.*

The centre of curvature is always on the *concave* side of the curve; also q at an inflexion or a cusp of the first species is ∞ or 0, and consequently the radius of curvature is either 0 or ∞.

Hence the four cases may be represented as in the annexed figures; in the two first the radius $= 0$; and in the two last the radius $= \infty$, it being a normal to the singular point.



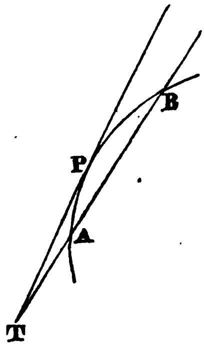
If the singular point of the involute is a cusp of the second species, then it appears, from describing the figure, that the corresponding point of the evolute is an inflexion whose tangent is a normal to the cusp.

24. *The different orders of inflexion.*

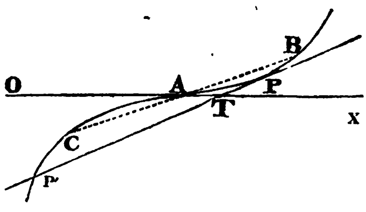
Explanation of terms.

A right line cannot cut a curve in more points than its equation has dimensions.

Let the curve be of the second degree; and let TAB cut it in the points A and B; suppose TAB to revolve round T so as to coincide with the tangent TP: hence it appears that at a point of simple contact we may suppose two consecutive points of the curve to coincide; and since a right line cannot meet a curve of the second degree in more than two points, the tangent cannot again meet the curve.



If the curve be of an higher degree, the tangent PT may meet it again in P′; suppose the line PTP′ to move so that P and P′ may coincide at A; hence at A it may be considered that three points coincide.

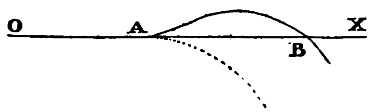


Or conceive a line, as CAB drawn through A, to revolve into the position OAX; then the three points A, B, and C co-

incide at A. It is obvious that A is an inflexion; it may be called a point of *simple* inflexion; which, therefore, may be defined to be a point at which the tangent meets the curve in *three* points.

Hence curves of the second degree do not admit of an inflexion: curves of the third degree admit of a simple inflexion; but the touching line OAX cannot meet the curve again.

If the curve be of an higher degree than the third, the tangent OAX may meet it again in B; and making the same supposition as before, viz. that AB move in such manner that B may coincide with A, at A four points of the curve may be considered as coinciding; and it is called a double point of inflexion, or a "point de serpentement," or of *undulation*.



If five points coincide at A, it is a *triple* inflexion; and if six, a *quadruple* inflexion, or a *double serpentement*.

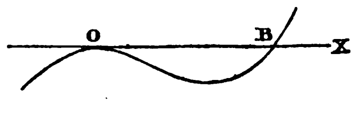
25. *Examples.*

In the following examples it is supposed that the origin has been transferred to the proposed point; or that the proposed point is the vertex.

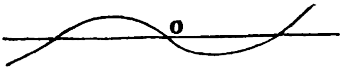
Ex. 1. Let $y = x^2$; then at the vertex, $y = 0$, and $x \times x = 0$; or two points only coincide, and consequently there is no inflexion.

Ex. 2. Let $y = x^3$; then at the vertex, $y = 0$, and $x \times x \times x = 0$; or x has three values each $= 0$, which shows that the axis x meets the curve in three points at the origin, and consequently there is a simple inflexion at that point.

Ex. 3. Let the equation be $y = x^3 - bx^2$; when $y = 0$, $x \times x \times (x - b) = 0$; or the curve touches X at the origin, and cuts it at a distance $x = b$: if in this equation, we suppose $b = 0$, then $y = x^3$, and three points coincide at 0, which is a point of simple inflexion.

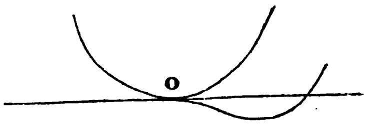


Ex. 4. Let the equation be $y = x^3 - b^2x$; when $y = 0$, $x.(x+b)(x-b)=0$, which shows that the curve cuts the axis in three points, which points coincide if $b = 0$; or, as before, $y = x^3$ has a simple inflexion at the origin.

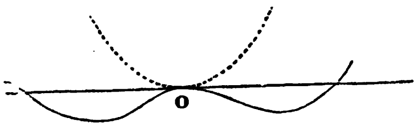


Ex. 5. $y = x^4$ has a double point of inflexion, or a serpentement at the origin.

Ex. 6. $y = x^4 - bx^3$; hence at 0, $(x-b)x.x.x=0$, or there is a simple inflexion at 0; and if $b = 0$, $y = x^4$, and there is a serpentement which is invisible.



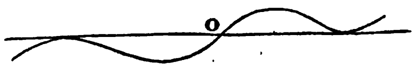
Ex. 7. Let $y = x^4 - (b^2 + c^2)x^2 + b^2c^2$. When $y=0$, $(x+b)(x-b)(x+c)(x-c) = 0$; or the curve cuts x in four points.



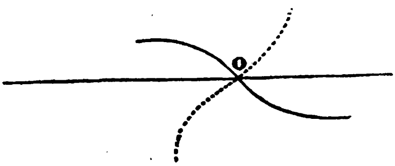
If $b = 0$, or the equation $y = x^4 - c^2x^2$, the curve meets x in two points at 0, or touches it at 0, and cuts it in two other points.

If c also $= 0$, the curve $y = x^4$ meets x in four points at 0, where there is a serpentement.

Ex. 8. Let $y = x^5 - (b^2 + c^2)x^3 + b^2c^2x$. When $y = 0, \ldots$ $x.(x - b)(x + b)(x - c)(x + c) = 0$; or the curve cuts x in five points, one of which is 0.



If $b = c$, or the equation is $y = x^5 - 2c^2x^3 + c^4x$, the two pairs coincide; or the curve touches the axis in two points.



If $b = 0$, there is a simple inflexion at 0, and the curve cuts x in two other points.

If $b = 0$ and $c = 0$, or the equation is $y = x^5$, there is a triple inflexion at 0, which is visible.

Cor. Inflexions of an odd order are visible; and those of an even order invisible.

26. *Miscellaneous Praxis.*

1. $y = x\sqrt{x-a}$ has a conjugate point at the origin.

2. $ay^3 - x^3y - bx^3 = 0$ has not a multiple point at the origin.

3. $\dfrac{y^4}{a^2} = y^2 - 3xy + 2x^2$ has a conjugate point at the origin.

4. $y - b = \dfrac{(x^2 - a^2)^{\frac{3}{2}}}{c^2}$. Cusps of the first species at $x = \pm a$, $y = b$, whose tangents are parallel to the axes.

5. $y^2 - (x-1)^2 x = 0$, or $y = (x-1)x^{\frac{1}{2}}$ has a double point at $x = 1$.

6. $y = b + (x-a)^{\frac{3}{5}}$. An inflexion at $x = a$, $y = b$, perpendicular to the axis.

7. $y = x^2 + (x-1)^{\frac{3}{5}}$. An inflexion at $x = 1$, $y = 1$ perpendicular to the axis.

8. $y = x^2 - x^{\frac{7}{3}} + x^3 = x^2(1 - x^{\frac{1}{3}} + x)$. The origin is a point of simple contact.

9. $y^3 - 2y^2x + x^3 = 0$, or $\left(\dfrac{y}{x}\right)^3 - 2\left(\dfrac{y}{x}\right)^2 + 1 = 0$. The curve consists of three right lines passing through the origin inclined to the axis at $\angle\, 45$ and $\tan.^{-1} \dfrac{1 \pm \sqrt{5}}{2}$.

10. $ay^2 - 2aby - x^3 - cx^2 + ab^2 = 0$. There is a double point at $y = b$; and the branches are equally inclined to the axis at $\tan.^{-1} \sqrt{\dfrac{c}{a}}$.

11. $a^2 - x^2 + (x-b)^2 = x^2y^2$ has a cusp of the first species at $x = \dfrac{a^2 + b^2}{2b}$.

12. $y^3 = ax^2 + x^3$ (Ch. 8. 13. Ex. 3). There is a cusp of the first species at the origin, and an inflexion at $x = -a$, which are perpendicular to the axis. There is a maximum

at $-x = \frac{8a^3}{27}$. The cusp and the inflexion have each a branch which has the same rectilinear asymptote.

13. $y = (x-2)\sqrt{\frac{x-9}{x}}$. There is a conjugate point at $x = 2$; a minimum at $x = -\frac{3}{2}$; and the equation of the asymptote is $y = \pm\left(x - \frac{13}{2}\right)$.

14. $y^4 - 2axy^2 + x^4 = 0$. (Vid. 5. 21. Ex. 5.) This curve consists of two *wings* issuing from the origin, where there is a cusp of the first species.

15. $\sqrt{x^2+y^2} = \frac{2axy}{\sqrt{x^2+y^2}}$. There is a quadruple point at the origin. (Vid. fig. 7. 14. Ex. 4.)

16. $x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0$.

Double points at $x = \pm a$, where $p = \pm\sqrt{\frac{4}{3}}$; and a third at $y = -a$, where $p = \pm\sqrt{\frac{2}{3}}$.

17. $y = (a+x)\sqrt{x}$; a conjugate point at $x = -a$.

18. $y^2 = b^2 + (x^2 - a^2)\sqrt{a^2 - 2x^2}$; four conjugate points at $x = \pm a$, $y = \pm b$.

19. $y = 1 + x\sqrt{x-1}$; a conjugate point at $y = 1$.

20. $y = (a+x)\sqrt{2rx - x^2}$; a conjugate point at $x = -a$.

21. $y^2 = \sqrt{x+1}\sqrt{x^2-1}$; a conjugate point at $x = -1$.

22. $y = 3\sqrt[3]{x-2}\,(1-x)^{\frac{3}{2}}$; a conjugate point at $x = 1$.

27. We shall conclude the Theory of Curves by giving the student some few general directions which are necessary to the complete discussion of any proposed curve. The axis of the curve is supposed to be X; but the same directions are applicable to the axis Y.

(1.) Observe whether the curve is symmetrical on the axis; and whether the origin is, or may be made, the centre of the curve. (7. 20.)

(2.) Solve the equation, if possible, with respect to one of the co-ordinates.

(3.) Observe from the changes of the signs (Alg. 311), the number of the positive and of the negative values of y when x is either positive or negative. If the roots of the equation are all possible, these changes will show the number and relative situation of the branches.

(4.) Find the points where the curve cuts the axes.

(5.) Decompose the equation, if possible, into factors, in order to ascertain if the curve is a compound curve.

(6.) Having cleared the equation of radicals, calculate the fluxional co-efficients p, q, and, if necessary, r. Observe whether there are any values of the variables which offer any peculiarity; if they render $p = \frac{0}{0}$, observe whether they also satisfy $\frac{du}{dx} = 0$, $\frac{du}{dy} = 0$.

(7.) Find the limits in the direction of the axes.

(8.) The number and position of the singular points being found, investigate the nature of each by transferring the origin to it, and developing y in an ascending series in terms of x.

(9.) Develope y in a descending series in terms of x; these will indicate the position and nature of the infinite branches.

Having thus ascertained the form of the curve, we may next proceed to investigate its properties.

In the first chapter of the second volume we shall select such curves as occur in physical science; or have become celebrated either from their properties, or from the use to which they were applied by the ancient geometers. For the history of the greater part of these curves, vid. Montucla's Histoire des Mathematiques.

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